18. Solvable groups

Recall that, if G is a group and g_1, g_2 are two elements of G, the ***commutator*** of g_1 and g_2 is $g_1 g_2 g_1^{-1} g_2^{-1}$. The ***commutator subgroup*** of G is the subgroup of G generated by the commutators of G.

Problem 18.1. Compute the commutator subgroups of the following groups. (This computation was on Problem Set 9; see if someone in your room has already done it.)

$$S_3, \qquad A_3, \qquad S_4, \qquad A_4, \qquad S_5 \quad \text{and} \quad A_5.$$

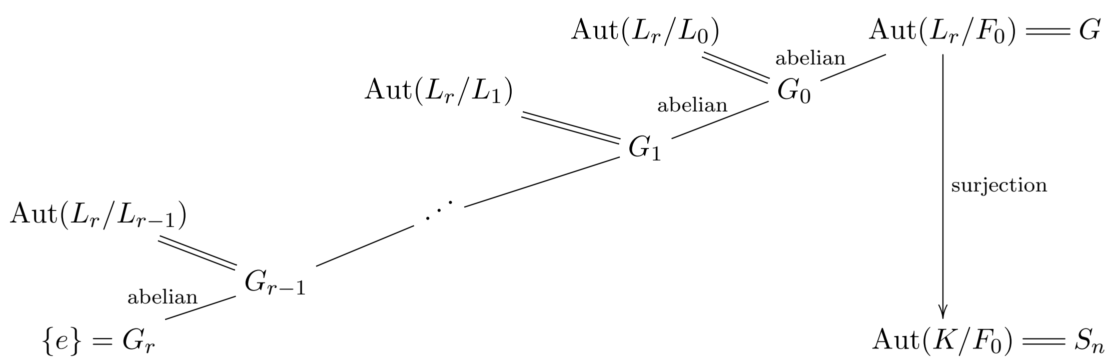
For a group G, we denote the commutator subgroup of G by DG. Let D^2G be the commutator subgroup of DG, let D^3G be the commutator subgroup of D^2G, and so forth. We define G to be ***solvable*** if there is some integer k for which $D^kG = \{e\}$.

Problem 18.2.
(1) Show that S_3 and S_4 are solvable.
(2) Show that S_n is not solvable for $n \geq 5$.

Problem 18.3. Suppose that we have a short exact sequence $1 \to G' \to G \to A \to 1$ where A is abelian and G' is solvable. Show that G is solvable.[3]

Problem 18.4. Suppose that $G \to H$ is a **surjective** homomorphism, and that G is solvable. Show that H is solvable.[4]

Now, remember the diagram from the end of our last worksheet:



Problem 18.5. Derive a contradiction!

You have proved the theorem:

Theorem: Let F_0 be a field, let $f(x)$ be a polynomial with coefficients in F_0 and let K be a splitting field for $f(x)$. Suppose that $\mathrm{Aut}(K/F_0)$ is S_n, for $n \geq 5$. Let F_0, F_1, F_2, …, be a sequence of fields such that each F_i is of the form $F_{i-1}[\sqrt[m_i]{a_{i-1}}]$ for some positive integer m_i and some a_{i-1} in F_{i-1}. Then $f(x)$ does not have roots in any of the F_i.

[3] In fact, a stronger result is true: If we have a short exact sequence $1 \to G_1 \to G_2 \to G_3 \to 1$ and G_1 and G_3 are solvable, then G_2 also is, but we don't need this.

[4] In fact, a stronger result is true: If we have a short exact sequence $1 \to G_1 \to G_2 \to G_3 \to 1$ and G_2 is solvable, then G_1 and G_3 are also solvable, but we only need the G_3 side.