## 2. The cubic formula

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| *Quando chel cubo con le cose appresso* | *When the cube with the cose beside it,* |
| *Se agguaglia à qualche numero discreto* | *equates itself to some other whole number,* |
| *Trouan dui altri differenti in esso. ...* | *find two others, of which it is the difference. ...* |

First three lines of a 25 line poem in which Tartaglia (1539) described the cubic formula. Translation by Friedrich Katscher (2011).

Let $\omega = \frac{-1+\sqrt{-3}}{2}$. Remember that

$$\omega^2 + \omega + 1 = 0 \text{ and } \omega^3 = 1.$$

Let $x^3 - e_1x^2 + e_2x - e_3$ be a cubic with roots $r_1$, $r_2$, $r_3$, which we want to find. So we have

$$e_1 = r_1 + r_2 + r_3 \qquad e_2 = r_1r_2 + r_1r_3 + r_2r_3 \qquad e_3 = r_1r_2r_3.$$

Define the following quantities:

$$\begin{array}{ccccccc} s_1 & = & r_1 & + & \omega r_2 & + & \omega^2 r_3 \\ s_2 & = & r_1 & + & \omega^2 r_2 & + & \omega r_3 \end{array} \tag{1}$$

Define the quadratic polynomial

$$y^2 - f_1y + f_2 := (y - s_1^3)(y - s_2^3). \tag{2}$$

The cubic formula can be described algorithmically as follows:

**Step 1:** Find the coefficients $f_1$ and $f_2$ of the quadratic (2).
**Step 2:** Use the quadratic formula to solve for $s_1^3$ and $s_2^3$.
**Step 3:** Take cube roots to find $s_1$ and $s_2$.
**Step 4:** Solve the linear equations (1), together with the equation $e_1 = r_1 + r_2 + r_3$, to find $r_1$, $r_2$ and $r_3$.

### Details of Step 1

**Problem 2.1.** Consider permuting the roots $r_1$, $r_2$ and $r_3$. (For example, you might switch $r_2$ and $r_3$, or cycle $r_1 \to r_2 \to r_3 \to r_1$.) How are the following quantities affected?

(1) $e_1$, $e_2$ and $e_3$
(2) $s_1$ and $s_2$.
(3) $s_1^3$ and $s_2^3$.
(4) $f_1$ and $f_2$.

**Problem 2.2.** Find formulas, in terms of $e_1$, $e_2$ and $e_3$, for the following quantities. I have prepared a cheat sheet of useful formulas (next page).

(1) $s_1s_2$
(2) $f_1$
(3) $f_2$

### Details of Step 4

**Problem 2.3.** Solve the linear equations (1), together with the equation $e_1 = r_1 + r_2 + r_3$, to find $r_1$, $r_2$ and $r_3$ in terms of $e_1$, $s_1$ and $s_2$.

### One last detail

As I described the algorithm, one takes cube roots twice, once to compute $s_1$ and once to compute $s_2$. In fact, one should only take one cube root, and then compute $s_2$ by the formula $s_2 = \frac{s_1s_2}{s_1}$. (Now you know why I had you compute $s_1s_2$ in Problem 2.2.) Otherwise, out of the 9 choices of which cube root to take, only 3 of the combinations will give correct solutions.

Cheatsheet: Some Useful Formulas

$$e_1 = r_1 + r_2 + r_3$$
$$e_2 = r_1 r_2 + r_1 r_3 + r_2 r_3 \qquad e_1^2 = r_1^2 + 2r_1r_2 + 2r_1r_3 + r_2^2 + 2r_2r_3 + r_3^2$$
$$e_3 = r_1 r_2 r_3 \qquad e_1e_2 = r_1^2r_2 + r_1^2r_3 + r_1r_2^2 + 6r_1r_2r_3 + r_1r_3^2 + r_2^2r_3 + r_2r_3^2$$
$$e_1^3 = r_1^3 + 3r_1^2r_2 + 3r_1^2r_3 + 3r_1r_2^2 + 6r_1r_2r_3 + 3r_1r_3^2 + r_2^3 + 3r_2^2r_3 + 3r_2r_3^2 + r_3^3$$

$$\begin{aligned}
s_1s_2 &= r_1^2 - r_1r_2 - r_1r_3 + r_2^2 - r_2r_3 + r_3^2\\
s_1^3 &= r_1^3 + 3\omega r_1^2r_2 + 3\omega^2r_1^2r_3 + 3\omega^2r_1r_2^2 + 6r_1r_2r_3 + 3\omega r_1r_3^2 + r_2^3 + 3\omega r_2^2r_3 + 3\omega^2r_2r_3^2\\
s_2^3 &= r_1^3 + 3\omega^2r_1^2r_2 + 3\omega r_1^2r_3 + 3\omega r_1r_2^2 + 6r_1r_2r_3 + 3\omega^2r_1r_3^2 + r_2^3 + 3\omega^2r_2^2r_3 + 3\omega r_2r_3^2\\
s_1^3 + s_2^3 &= 2r_1^3 - 3r_1^2r_2 - 3r_1^2r_3 - 3r_1r_2^2 + 12r_1r_2r_3 - 3r_1r_3^2 + 2r_2^3 - 3r_2^2r_3 - 3r_2r_3^2 + 2r_3^3
\end{aligned}$$