## 4. GROUPS, ORBITS AND STABILIZERS

*A set of symbols 1, $\alpha$, $\beta$,...all of them different, and such that the product of any two of them ... belongs to the set, is said to be a group. ...* Cayley, 1854

We will want to be able to talk about a group without saying what it is a group of permutations of. Here is our definition:

**Definition:** A ***group*** is a set $G$ with a binary operation $\times$ such that

(1) There is an element 1 of $G$, called the ***identity***, such that $1 \times g = g \times 1 = g$ for all $g$ in $G$.
(2) For every $g$ in $G$, there is an element $g^{-1}$, called the ***inverse of*** $g$ such that $g \times g^{-1} = g^{-1} \times g = 1$.
(3) Multiplication is ***associative*** meaning that, for any three groups elements $f$, $g$ and $h$, we have $f \times (g \times h) = (f \times g) \times h$.

Note that, for a group of permutations, associativity is a theorem (Problem 3.1) whereas, for a group in general, it is an axiom.

**Problem 4.1.** Prove or disprove and salvage if possible:

(1) The integers modulo $n$ are a group, where the group operation is $+$.
(2) The integers modulo $n$ are a group, where the group operation is $\times$.

We want to be able to start with an abstract group $G$ and turn it into a set of permutations of a set $X$. We do this with the notion of an ***action***:

**Definition:** Let $G$ be a group and $X$ a set. A ***action*** of $G$ on $X$ is a binary operation $G \times X \to X$ such that

(1) For all $x \in X$, we have $1 \times x = x$ and
(2) For all $f$ and $g$ in $G$ and all $x \in X$, we have

$$(f \times g) \times x = f \times (g \times x).$$

Note that it is okay to have $g \times x = x$ for $g \neq 1$.

We repeat the definitions of ***orbit*** and ***stabilizer*** in this context: If a group $G$ acts on a set $X$, and $x$ is an element of $X$, then the ***orbit of*** $x$, written $Gx$, is $\{gx \ : \ g \in G\}$. The ***stabilizer of*** $x$ is $\{g \in G : gx = x\}$.

**Problem 4.2.** Let $G$ be the set of rotational symmetries of a cube, with multiplication being composition. Describe the orbit and the stabilizer of a point $x$ in each of the following sets:

(1) The set of faces of the cube.
(2) The set of edges of the cube.
(3) The set of vertices of the cube.

**Problem 4.3.** Let $G$ be the group of rotational symmetries of a square pyramid. Describe the orbits of $G$ acting on

(1) The set of faces of the pyramid.
(2) The set of edges of the pyramid.
(3) The set of vertices of the pyramid.

**Problem 4.4.** Let $X$ be a set and let $G$ be a finite group acting on $X$. For $x \in X$, show that

$$\#(G) = \#\mathrm{Stab}(x) \cdot \#(Gx).$$

**Problem 4.5.** Let $G$ be a finite group acting on a set $X$. Let $x$ be an element of $X$ and let $Gx$ be the orbit of $x$. Show that $\#(Gx)$ divides $\#(G)$.

**Alternate notations:** The group multiplication $g \times h$ may also be written $gh$. When $G$ acts on $X$, we may also write $g \times x$ as $g(x)$ or $gx$.