6. Characters of groups

You also get dramatic advances when you spot that you can leave out part of the problem. Algebra, for instance . . . derives from the realization that you can leave out all the messy, intractable numbers.
Douglas Adams, *The Salmon of Doubt*, 2002

Let G be a group. Recall that $\mathbb{C}$ is the complex numbers and $\mathbb{C}^*$ is nonzero complex numbers.

A ***character*** is a map $\chi : G \to \mathbb{C}^*$ such that, for all g_1 and g_2 in G, we have

$$\chi(g_1 \times g_2) = \chi(g_1)\chi(g_2).$$

Problem 6.1. Let χ be a character of G and let g_1 and g_2 in G. Show that

$$\chi(g_1 g_2 g_1^{-1} g_2^{-1}) = 1.$$

An element of G of the form $g_1 g_2 g_1^{-1} g_2^{-1}$ is called a ***commutator***.

Problem 6.2. Let χ be a character of G. The ***kernel of*** χ is the set of g in G such that $\chi(g) = 1$. Show that the kernel of χ is always a subgroup of G.

Problem 6.3. Let D be the group of symmetries (reflections and rotations) of the square. Find 4 characters of D. (One of them will be the trivial character, $\chi(g) = 1$ for all $g \in D$.)