> *Throughout the book, I'll suggest some problems for you to play with. You might be tempted to skip them ... thinking, well, they've probably been done by someone else, so what's the point? ... But so what? Do them for the fun of it! That's how you learn the knack of doing things ...*
>
> Richard Feynman, *Lectures on Computation*, 1984

Let S_n act on the polynomial ring $\mathbb{C}[r_1, \ldots, r_n]$ by reordering the variables. Problem 7.4 generalizes Problems 7.1, 7.2 and 7.3. Perhaps some of you would prefer to do the general problem first.

Problem 7.1. Set

$$\Delta(r_1, r_2, \ldots, r_n) = \prod_{1 \leq i < j \leq n} (r_i - r_j).$$

(1) Show that, for all $w \in S_n$, we have
$$w\Delta = \pm\Delta.$$
(2) Define $\epsilon(w) = \frac{w\Delta}{\Delta}$. Show that ϵ is a character of S_n.

Problem 7.2. Let $\omega = \frac{-1+\sqrt{-3}}{2}$. Set

$$a(r_1, r_2, r_3) = r_1 + \omega r_2 + \omega^2 r_3.$$

(1) Show that the stabilizer of a^3 in S_3 is a group A_3 with 3 elements.
(2) For $g \in A_3$, define $\alpha(g) = \frac{ga}{a}$. Show that α is a character of A_3,

Problem 7.3. Let $\omega = \frac{-1+\sqrt{-3}}{2}$. Set

$$b(r_1, r_2, r_3, r_4) = (r_1 r_2 + r_3 r_4) + \omega(r_1 r_3 + r_2 r_4) + \omega^2(r_1 r_4 + r_2 r_3).$$

(1) Show that the stabilizer of b in S_4 is a group K with 4 elements.
(2) Show that the stabilizer of b^3 is a group A_4 with 12 elements.
(3) For $g \in A_4$, define $\beta(g) = \frac{gb}{b}$. Show that β is a character of A_4, and K is the kernel of β.

Problem 7.4. Let $F(r_1, r_2, \ldots, r_n)$ be a nonzero polynomial, and let k be a positive integer.

(1) Show that, for all $g \in \mathrm{Stab}(F^k)$, we have $g(F) = \chi(g)F$ for some scalar $\chi(g)$ in $\mathbb{C}^*$.
(2) Show that χ is a character of $\mathrm{Stab}(F^k)$.