8. Characters of the symmetric and alternating groups

Lemmas do the work in mathematics. Theorems, like management, just take the credit.
Paul Taylor, Practical Foundations of Mathematics, 1999

Recall that the symmetric group S_n is the group of all permutations of $\{1, 2, \ldots, n\}$. In this worksheet, we will determine the characters of S_n, and of its close relative A_n.

In Problem 7.1, we defined a character $\epsilon : S_n \to \{\pm 1\}$ by

$$\epsilon(g) = \frac{g\Delta(r_1, \ldots, r_n)}{\Delta(r_1, \ldots, r_n)} \quad \text{where} \quad \Delta(r_1, r_2, \ldots, r_n) = \prod_{1 \leq i < j \leq n} (r_i - r_j).$$

We define the ***alternating group***, A_n, to be the set of permutations w in A_n with $\epsilon(w) = 1$.

One tool that we will want is the notion of a conjugacy class.

Problem 8.1. Let G be a group and let g_1 and g_2 be two elements of G. We say that g_1 is ***conjugate to*** g_2 if there is some h in G with $g_1 = hg_2h^{-1}$.

(1) Show that "conjugate to" is an equivalence relation. The equivalence classes of this equivalence relation are called ***conjugacy classes***.
(2) List the conjugacy classes of S_3 and S_4.
(3) Show that, if g_1 and g_2 are conjugate, and χ is a character of G, then $\chi(g_1) = \chi(g_2)$.

A ***transposition*** is an element of S_n which fixes $n-2$ of the numbers from $\{1, 2, \ldots, n\}$ and switches the other 2.

Problem 8.2. Show that all the transpositions in S_n form a single conjugacy class.

The following problem was on your homework, find someone in your group who did it:

Problem 8.3. Show that every element in S_n is a product of transpositions.

Problem 8.4. Let $n \geq 2$ and let χ be a character of S_n.

(1) Show that there is a single complex number, e, such that $\chi(t) = e$ for every transposition t.
(2) Show that $e = \pm 1$.
(3) If $e = 1$, show that $\chi(g) = 1$ for all permutations g. If $e = -1$, show that $\chi(g) = \epsilon(g)$.

We have thus shown that **every character of S_n is either the trivial character, or the character** ϵ. We now turn to the group A_n. In Problems 7.2 and 7.3, you constructed characters of A_3 and A_4. Our goal now is to show that, **for $n \geq 5$, the group A_n only has the trivial character**.

A permutation c in S_n is called a 3-***cycle*** if there are three distinct elements i, j and k of $\{1, 2, \ldots, n\}$ such that $t(i) = j$, $t(j) = k$, $t(k) = i$ and $t(\ell) = \ell$ for $\ell \neq i, j, k$.

Problem 8.5. Let $n \geq 3$.

(1) Show that every 3-cycle in S_n is in A_n.
(2) Show that every element in A_n is a product of 3-cycles.

Problem 8.6. Let $n \geq 3$, let χ be a character of A_n and let c be a 3-cycle.

(1) Show that $\chi(c)^3 = 1$.
(2) If $n \geq 5$, show that c and c^{-1} are conjugate to each other in A_n.
(3) If $n \geq 5$, show that $\chi(c) = 1$.

Problem 8.7. Let $n \geq 5$ and let χ be a character of A_n. Show that $\chi(g) = 1$ for all g in A_n.