9. Unsolvability of the quintic – first version

> *Perhaps it will not be so difficult to prove, with all rigor, the impossibility for the fifth degree.*
>
> Karl Freidrich Gauss, 1799

The aim of this worksheet is to prove, in a certain sense, that there is no universal formula for the roots of a quintic equation. We first lay out exactly what we will, and will not, prove.

Let L be the field of formal rational functions in the variables r_1, r_2, r_3, r_4 and r_5. Let $K \subset L$ be the field of symmetric rational functions in these variables. Thus, if we start with a general quintic $x^5 - e_1 x^4 + e_2 x^3 - e_3 x^2 + e_4 x + e_5 = (x - r_1)(x - r_2) \cdots (x - r_5)$, then the coefficients $e_1, \ldots, e_5$ are in K.

> **Theorem:** Suppose we compute a sequence of elements of L, starting with $e_1, e_2, \ldots, e_5$, and applying the operators $+$, $-$, $\times$, $\div$ and $\sqrt[n]{\ }$. **Every time that we apply the operator** $\sqrt[n]{\ }$**, we insist that the output of the n-th root still lies in** L. Then we can never obtain the elements $r_1, r_2, \ldots, r_5$ of L.

We note what we are **not** proving. We are not considering the possibility of formulas where the n-th root leaves the realm of rational functions. Even more so, we cannot take any particular quintic with rational coefficients and conclude that its roots cannot be expressed in terms of $+$, $-$, $\times$, $\div$ and $\sqrt[n]{\ }$. In the second half of the course, we will address these issues.

But this Theorem is already strong enough to rule out a formula that looks anything like the quadratic, cubic or quartic formulas. We now turn to the proof.

Define $F \subset L$ to be the field of functions which are fixed by the alternating group $A_5 \subset S_5$. We will show that the operations $+$, $-$, $\times$, $\div$ and $\sqrt[n]{\ }$ cannot get us out of F.

Problem 9.1. Suppose that $f(r_1, r_2, \ldots, r_5)$ is in F and $\sqrt[n]{f}$ is in L. Show that $\sqrt[n]{f}$ is in F.

Problem 9.2. Suppose that $f(r_1, r_2, \ldots, r_5)$ is in F, that $\sqrt[n]{f}$ is in L and that $f \neq 0$. Show that $\chi(w) := \frac{w(\sqrt[n]{f})}{\sqrt[n]{f}}$ is a character of A_5. (Some people found this phrasing confusing. Here is an alternate phrasing which I would find more confusing, but maybe you won't: Let $f \in L$ be nonzero and suppose that there exists a function $h \in F$ with $h^n = f$. Define $\chi(w) = \frac{w(h)}{h}$.)

But Problem 8.7 shows that the only character of A_5 is the character which always takes the value 1.

Problem 9.3. Suppose that $f(r_1, r_2, \ldots, r_5)$ is in F and $\sqrt[n]{f}$ is in L. Show that $\sqrt[n]{f}$ is in F

Problem 9.4. Explain why we have proved the Theorem!