Halfway reflection point – due Monday July 26.

Each of the four problems in this problem set encourages you to write up a major result in detail. The steps given are meant as a guide, but the main point is to prove the big result. I would suggest choosing the ones that interest you, and doing them well.

Congratulations on reaching the halfway point!

Problem 1. Let G be a finite group and let X be a set on which G acts. For $x \in X$, let Gx be the ***orbit*** of x, defined as $\{g(x) : g \in G\}$ and let $\mathrm{Stab}(x)$ be the ***stabilizer*** of x, defined as $\{g : g(x) = x\}$.

(1) Prove the orbit-stabilizer theorem: $\#(G) = \#(Gx)\ \#(\mathrm{Stab}(x))$.
(2) Let $f(r_1, \ldots, r_n)$ be a rational function of r_1, r_2, …, r_n. Show that the number of different functions we can get by permuting the r_i is a divisor of $n!$.

On the problem sets, we showed that, if $n \geq 5$ and $n > m$, then f cannot take on m values. Let's do something a bit easier: Let's show that, if $n \geq 5$, then f cannot take on exactly 3 values. Suppose, for the sake of contradiction, that $n \geq 5$ and that we have a rational function $f(r_1, \ldots, r_n)$ which takes on precisely 3 values, f_1, f_2 and f_3, as we permute the r_i.

(3) Show that every permutation in A_n permutes $\{f_1, f_2, f_3\}$ by a permutation in A_3. Hint: Consider $S_n \longrightarrow S_3 \xrightarrow{\epsilon} \{\pm 1\}$.
(4) Show that every permutation in A_n fixes $\{f_1, f_2, f_3\}$. Hint: Consider $A_n \longrightarrow A_3 \xrightarrow{\alpha} \{1, \omega, \omega^2\}$, where α is a nontrivial character of A_3.
(5) Deduce a contradiction.

Problem 2. Let L be the field of formal rational functions in the variables r_1, r_2, r_3, r_4 and r_5. Let $K \subset L$ be the field of symmetric rational functions in these variables. Thus, if we start with a general quintic $x^5 - e_1x^4 + e_2x^3 - e_3x^2 + e_4x - e_5 = (x - r_1)(x - r_2)\cdots(x - r_5)$, then the coefficients e_1, …, e_5 are in K. In this problem, you will show:

Theorem: Suppose we compute a sequence of elements of L, starting with e_1, e_2, …, e_5, and applying the operators $+$, $-$, $\times$, $\div$ and $\sqrt[n]{}$. **Every time that we apply the operator $\sqrt[n]{}$, we insist that the output of the n-th root still lies in L.** Then we can never obtain the elements r_1, r_2, …, r_5 of L.

Let F be the subgroup of L consisting of functions which are invariant for A_5.

(1) Suppose that $f \in F_{\neq 0}$ and $\sqrt[n]{f} \in L$. Define $\chi(g) = \frac{g(\sqrt[n]{f})}{\sqrt[n]{f}}$. Show that χ is a character of A_5.
(2) Show that the group A_5 has no nontrivial characters.
(3) Show that, if $f \in F$ and $\sqrt[n]{f} \in L$, then $\sqrt[n]{f} \in F$.
(4) Prove that starting with e_1, e_2, …, e_5, and applying the operators $+$, $-$, $\times$, $\div$ and $\sqrt[n]{}$ as described above, we will never obtain any function not in F.

Problem 3. Throughout this problem k be a field. If you are uncomfortable with that level of generality, let k be $\mathbb{Q}$.

(1) Let $f(x)$ and $g(x)$ be nonzero polynomials in $k[x]$. Show that $\deg(fg) = \deg(f) + \deg(g)$.
(2) Let k be a field and let $a(x)$ and $b(x)$ be polynomials in $k[x]$, with $b(x) \neq 0$. Show that there are $q(x)$ and $r(x)$ with $a(x) = b(x)q(x) + r(x)$ and either $\deg r(x) < \deg b(x)$ or $r(x) = 0$.
(3) Let k be a field and let $a(x)$ and $b(x)$ be polynomials in $k[x]$. Show that the following conditions are equivalent:
   (a) There is no polynomial $d(x)$ with $\deg d > 0$ dividing both $a(x)$ and $b(x)$.
   (b) There are polynomials $u(x)$ and $v(x)$ such that $a(x)u(x) + b(x)v(x) = 1$.
(4) Let $p(x)$ be an irreducible polynomial in $k[x]$, and let $q(x)$ be a nonzero element of $k[x]_{p(x)}$. Show that $q(x)$ is a unit of $k[x]_{p(x)}$.

Problem 4. I feel guilty that we never learned the quartic formula. So here is a problem about it. Consider the quartic polynomial

$$x^4 - e_1x^3 + e_2x^2 - e_3x + e_4 = (x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

We let S_4 act on polynomials in r_1, r_2, r_3 and r_4. Set $\ell = r_1 - r_2 + r_3 - r_4$.

(1) What is the stabilizer of ℓ in S_4? What is the orbit of ℓ?
(2) What is the stabilizer of ℓ^2 in S_4? What is the orbit of ℓ^2?
(3) Show that ℓ^2, and the other elements of its orbit, can be computed from e_1, e_2, e_3 and e_4, using the operations $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$. You may use the Fundamental Theorem of Symmetric Functions (Problem Set 2) and the cubic formula (Worksheet 2).
(4) Show how to compute the roots r_1, r_2, r_3 and r_4 in terms of ℓ, the other elements in the orbit of ℓ, and e_1.

Problem 5. What is a question or conjecture that you have about the material so far?