Problem Set 1 – due Thursday July 8.

There will definitely be more problems on my problem sets than most people will want to do. Take a look at them, see which ones appeal to you, and find fellow students to work with!

Problem 1. Let r_1, r_2, r_3 be the roots of the cubic $x^3 - 2x^2 + 3x - 5 = 0$. Compute

(1) $r_1^2 + r_2^2 + r_3^2$.
(2) $r_1^3 + r_2^3 + r_3^3$.

The way to do this does **not** involve finding closed formulas for the roots of this cubic.

Problem 2. Let α be one the roots of $x^3 + x^2 - 2x - 1 = 0$. Show that $\alpha^2 - 2$ is also a root of this polynomial. Can you express the third root of this polynomial in terms of α as well?

Problem 3. (1) Let ω be a primitive cube root of unity, meaning that $\omega^3 = 1$ and $\omega \neq 1$. Solve the following equations for x_0, x_2, x_2 in terms of a_0, a_1, a_2:

$$\begin{array}{ccccccc} x_0 & + & x_1 & + & x_2 & = & a_0 \\ x_0 & + & \omega x_1 & + & \omega^2 x_2 & = & a_2 \\ x_0 & + & \omega^2 x_1 & + & \omega x_2 & = & a_2 \end{array}$$

(2) Let ζ be a primitive fifth root of unity, meaning that $\zeta^5 = 1$ and $\zeta \neq 1$. Solve the following equations for y_0, y_1, y_2, y_3, y_4 in terms of b_0, b_1, b_2, b_3, b_4:

$$\begin{array}{ccccccccccc} x_0 & + & x_1 & + & x_2 & + & x_3 & + & x_4 & = & c_0 \\ x_0 & + & \zeta x_1 & + & \zeta^2 x_2 & + & \zeta^3 x_3 & + & \zeta^4 x_4 & = & c_1 \\ x_0 & + & \zeta^2 x_1 & + & \zeta^4 x_2 & + & \zeta x_3 & + & \zeta^3 x_4 & = & c_2 \\ x_0 & + & \zeta^3 x_1 & + & \zeta x_2 & + & \zeta^4 x_3 & + & \zeta^2 x_4 & = & c_3 \\ x_0 & + & \zeta^4 x_1 & + & \zeta^3 x_2 & + & \zeta^2 x_3 & + & \zeta x_4 & = & c_4 \end{array}$$

Problem 4. We consider polynomials $f(q_1, q_2, q_3)$ in three variables q_1, q_2, q_3. Given a permutation σ of $\{1, 2, 3\}$, we will say that it ***fixes*** f if the formula f is unchanged when we permute the variables by σ. For example, $q_1 + q_2 q_3$ is fixed by the permutation which switches 2 and 3, but not the permutation which switches 1 and 2.

Can you give polynomials f which are fixed by the following permutations, and only these? Explain why or why not.

(a) $123 \mapsto 123,\ 213$
(b) $123 \mapsto 123,\ 231,\ 312$
(c) $123 \mapsto 123,\ 213,\ 132.$

Problem 5. We consider polynomials $f(r_1, r_2, r_3, r_4)$ in four variables r_1, r_2, r_3, r_4. The ***orbit*** of a polynomial is all polynomials we can obtain by permuting the variables. For which of the following numbers n can you find a polynomial f whose orbit has size n?

1 2 3 4 5 6 7 8 9 10.