Problem Set 2 – due Monday July 12.

The fundamental theorem of symmetric polynomials

Let $r_1, r_2, \dots, r_n$ be formal variables. Let $e_1, e_2, \dots, e_n$ be polynomials in the r's, defined as the coefficients of

$$(t - r_1)(t - r_2)\cdots(t - r_n) = t^n - e_1 t^{n-1} + e_2 t^{n-2} - \cdots + (-1)^n e_n.$$

The fundamental theorem of symmetric polynomials states

Theorem. *If f is a polynomial in $r_1, r_2, \dots, r_n$ which is fixed by all permutations of 1, 2, $\dots$, n then f is a polynomial in $e_1, e_2, \dots, e_n$.*

In Problem Set 1, Problem 1, you worked out some examples of the fundamental theorem of symmetric polynomials. Now, you will prove it.

Choose positive real numbers $w_1 > w_2 > \cdots > w_n$. Given a vector $(d_1, d_2, \dots, d_n)$ of nonnegative integers, define the ***weight*** of $(d_1, \dots, d_n)$ to be $\sum_{i=1}^{n} w_i d_i$. We define the ***weight of a monomial*** $r_1^{d_1} r_2^{d_2} \cdots r_n^{d_n}$ to be the weight of $(d_1, \dots, d_n)$. For a nonzero polynomial f, we define the ***leading monomial of*** f to be the monomial of largest weight whose coefficient in f is nonzero.

Problem 1. For any positive real number W, show that there are only finitely many vectors $(d_1, d_2, \dots, d_n)$ of nonnegative integers with weight $\leq W$.

Problem 2. Suppose that $f(r_1, \dots, r_n)$ is symmetric and let $r_1^{d_1} r_2^{d_2} \cdots r_n^{d_n}$ be its leading monomial. Show that $d_1 \geq d_2 \geq \cdots \geq d_n$.

Problem 3. Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be nonnegative integers. Show that the leading monomial of $e_1^{d_1-d_2} e_2^{d_2-d_3} \cdots e_{n-1}^{d_{n-1}-d_n} e_n^{d_n}$ is $r_1^{d_1} r_2^{d_2} \cdots r_n^{d_n}$.

We now give an algorithm to express any symmetric function as a polynomial in $e_1, e_2, \dots,$ e_n. Let f be a nonzero symmetric polynomial, and let its leading monomial be $r_1^{d_1} \cdots r_n^{d_n}$ with coefficient c. Let $f' = f - c e_1^{d_1-d_2} e_2^{d_2-d_3} \cdots e_{n-1}^{d_{n-1}-d_n} e_n^{d_n}$. If f' is 0, we are done. If not, let f' have leading monomial $r_1^{d'_1} \cdots r_n^{d'_n}$ with coefficient c' and let $f'' = f' - c' e_1^{d'_1-d'_2} \cdots e_{n-1}^{d'_{n-1}-d'_n} e_n^{d'_n}$. We continue in this manner; the algorithm stops if we get to the zero polynomial.

Problem 4. Show that this algorithm does, eventually, stop. Hint: Look at Problem 1.

Other questions

Problem 5. A permutation t in the symmetric group S_n is called a ***transposition*** if there are two distinct elements i and j of $\{1, 2, \dots, n\}$ such that $t(i) = j$, $t(j) = i$ and $t(k) = k$ for $k \neq i, j$. Show that every permutation can be written as a composition $t_1 \circ t_2 \circ \cdots \circ t_N$ for some N and some list of transpositions $t_1, t_2, \dots, t_N$.

Problem 6. Let η be a primitive n-th root of unity, meaning that $\eta^n = 1$ and $\eta^k \neq 1$ for any $1 \leq k \leq n-1$. Let $z_0, z_1, \dots, z_{n-1}$ be formal variables and set

$$c_j = \sum_{i=0}^{n-1} \eta^{ij} z_i.$$

Express the z_i in terms of the c_j. (See Problem Set 1, Problem 3 for examples.)