Problem Set 3 – due Thursday July 15.

Practice with the group axioms

The point of this problem is to practice proving things directly from the axioms of a group (which appeared on worksheet 4).

Problem 1. Let G be a group. Show that

(1) G has only one identity, meaning there is only one element 1 of G with the property that $1g = g1 = g$ for all g in G.
(2) Given g in G, there is only one inverse to g, meaning one element g^{-1} such that $gg^{-1} = g^{-1}g = 1$.
(3) For any g and h in G, we have $(gh)^{-1} = h^{-1}g^{-1}$.

Group homomorphisms, kernels and normal subgroups

This problem introduces some tools which we will need for any serious work with groups, although we will manage to avoid them for the first few weeks.

Let G and H be two groups. A ***homomorphism*** is a function $\phi : G \to H$ with the property that

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$

for all g_1 and g_2 in G. So a character is a homomorphism from G to $\mathbb{C}^*$.

Problem 2. Let the group S_4 act on polynomials in 4-variables x_1, x_2, x_3, x_4 by permuting the variables. Set $y_1 = x_1x_2 + x_3x_4$, $y_2 = x_1x_3 + x_2x_4$, $y_3 = x_1x_4 + x_2x_3$. Show that the action of S_4 on the polynomials y_1, y_2, y_3 gives a homomorphism $S_4 \to S_3$. There is very little to actually do here; the point is to understand what the words mean.

Problem 3. Let G and H be two groups and let $\phi : G \to H$ be a homomorphism.

(1) Show that $\phi(1_G) = 1_H$, where 1_G is the identity of G and 1_H is the identity of H.
(2) Show that $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

The ***kernel of*** ϕ is the set of $g \in G$ such that $\phi(g) = 1$.

Problem 4. Describe the kernel of the homomorphism $S_4 \to S_3$ from Problem 2.

Problem 5. Let G and H be two groups, let $\phi : G \to H$ be a homomorphism and let K be the kernel of ϕ.

(1) Show that K is a subgroup of G. This means that
    (a) The identity, 1_G, is in K.
    (b) If k is in K, then k^{-1} is in K
    (c) If k_1 and k_2 are in K, then k_1k_2 is in K.
(2) Show that, if k is in K and g is in G, then gkg^{-1} is in K.

Problem 6. A subgroup N of a group G is called ***normal*** if, whenever $n \in N$ and $g \in G$, we have $gng^{-1} \in N$. So the last part of Problem 5 showed that the kernel of a homomorphism is always normal.

(1) Show that the only normal subgroups of S_5 are $\{e\}$, A_5 and S_5. This can be done by cleverly applied force, though probably not by pure brute force.
(2) (**Challenge**) Let $n \geq 5$. Show that the only normal subgroups of S_n are $\{e\}$, A_n and S_n.