Problem Set 5 – due Thursday July 22.

More group theory

Problem 1. This problem continues the ideas from Problem Set 4, Problem 4.

(1) Suppose that $\#(X)$ is prime, and that G acts on X with a single orbit. Show that either every two elements of X are switchable, or else that no element is switchable with any element other than itself.
(2) Let p be a prime and let G be a group of permutations of $\{1, 2, \ldots, p\}$ which has a single orbit and contains a transposition. Show that G is the entire group S_p.
(3) Give an example of a group of permutations of $\{1, 2, 3, 4\}$ which has a single orbit and contains a transposition, but is not S_4.

Problem 2. A group G is called ***abelian*** if $g_1 g_2 = g_2 g_1$ for all g_1 and g_2 in G.

(1) Show that, if G is abelian and A is a subgroup of G, then A is abelian.
(2) Show that, if G is abelian, and $G \to B$ is a surjective group homomorphism, then B is abelian.
(3) Show that, if G is abelian, and H is any subgroup of G, then H is normal.

Problem 3. Let G be a group and let N be a normal subgroup. In this problem, we will define the "quotient group" G/N. The elements of G/N will be the left cosets gN.

(1) We define the product of the cosets gN and hN to be the coset ghN. Show that this is well-defined, meaning that, if $gN = g'N$ and $hN = h'N$, then $ghN = g'h'N$.
(2) Show that this product makes G/N into a group.

Preparing for field theory

Problem 4. Write the following numbers in the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ with a, b and c integers

$$1,\ 1 + \sqrt[3]{2},\ (1 + \sqrt[3]{2})^2,\ (1 + \sqrt[3]{2})^3.$$

Problem 5. The different parts of this problem are connected.

(1) Find polynomials $q(t)$ and $r(t)$ such that $\deg r(t) \leq 1$ and

$$t^3 - 2 = (t^2 + 4t + 5)q(t) + r(t).$$

(2) Find polynomials $x(t)$ and $y(t)$ such that

$$(t^3 - 2)x(t) + (t^2 + 4t + 5)y(t) = 1.$$

(3) Find rational numbers a, b and c such that

$$a + b\sqrt[3]{2} + c\sqrt[3]{4} = \frac{1}{\sqrt[3]{4} + 4\sqrt[3]{2} + 5}.$$