Problem Set 6 – due Thursday July 29.

Fields

Problem 1. Prove the following results directly from the axioms of a field (Worksheet 10):

(1) If k is a field and x is an element of k, then $x0 = 0x = 0$.
(2) If k is a field and x and y are elements of k with $xy = 0$, then $x = 0$ or $y = 0$.

Problem 2. Write out addition and multiplication tables for the following fields:

$$(1)\ \mathbb{Z}_2[x]_{x^2+x+1} \qquad (2)\ \mathbb{Z}_2[y]_{y^3+y+1} \qquad (3)\ \mathbb{Z}_3[z]_{z^2+1}.$$

Problem 3. Let K and L be two fields. A ***isomorphism*** between K and L is a bijection $\phi : K \to L$ such that $\phi(x+y) = \phi(x)+\phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$. For each of the following pairs of fields, determine whether or not they are isomorphic:
(1) $\mathbb{Q}[x]_{x^2+3}$ and $\mathbb{Q}[y]_{y^2+y+1}$. (2) $\mathbb{Q}[x]_{x^2-2}$ and $\mathbb{Q}[y]_{y^2+2}$ (3) $\mathbb{Z}_5[x]_{x^2-2}$ and $\mathbb{Z}_5[y]_{y^2+2}$.

Linear Algebra

Problem 4. Find a solution in $\mathbb{Q}^4$, other than $(0, 0, 0, 0)$, to this system of equations:

$$\begin{array}{ccccccccc} 12x_1 & + & 3x_2 & + & x_3 & & & = & 0 \\ 15x_1 & + & 4x_2 & + & x_3 & & & = & 0 \\ 19x_1 & + & 5x_2 & + & x_3 & + & x_4 & = & 0 \end{array}$$

Problem 5. (**The Key Lemma of Linear Algebra**:) Let k be a field and let $m < n$ be nonnegative integers. Let A_{ij}, for $1 \le i \le m$ and $1 \le j \le n$, be mn elements of k. In this problem, you will prove that the system of equations

$$\begin{array}{ccccccccccc} A_{11}x_1 & + & A_{12}x_2 & + & \cdots & + & A_{1(n-1)}x_{n-1} & + & A_{1n}x_n & = & 0 \\ A_{21}x_1 & + & A_{22}x_2 & + & \cdots & + & A_{2(n-1)}x_{n-1} & + & A_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \cdots & & \vdots & & \vdots & & \\ A_{(m-1)1}x_1 & + & A_{(m-1)2}x_2 & + & \cdots & + & A_{(m-1)(n-1)}x_{n-1} & + & A_{(m-1)n}x_n & = & 0 \\ A_{m1}x_1 & + & A_{m2}x_2 & + & \cdots & + & A_{m(n-1)}x_{n-1} & + & A_{mn}x_n & = & 0 \end{array}$$

has a solution in k^n, **other than** $(0, 0, \ldots, 0)$. Our proof is by induction on m.

(1) Do the base case, $m = 1$. Or, if you like, do the base case $m = 0$ and then make sure your inductive proof covers the case $m = 1$.

So, now let $m > 1$ and suppose that we have proved the result for all smaller values of m.

(2) Show that the result is true if all of the A_{ij} are 0.

So we may assume that at least one A_{ij} is nonzero; without loss of generality, let $A_{mn} \neq 0$. Make $m - 1$ equations in the $n - 1$ variables x_1, x_2, $\ldots$, x_{n-1} by subtracting A_{in}/A_{mn} times the last equation from the i-th equation. In other words, the new equations are

$$\begin{array}{ccccccc} (A_{11} - \frac{A_{1n}A_{m1}}{A_{mn}})x_1 + & (A_{12} - \frac{A_{1n}A_{m2}}{A_{mn}})x_2 + & \cdots & + & (A_{1(n-1)} - \frac{A_{1n}A_{m(n-1)}}{A_{mn}})x_{n-1} & = & 0 \\ (A_{21} - \frac{A_{2n}A_{m1}}{A_{mn}})x_1 + & (A_{22} - \frac{A_{2n}A_{m2}}{A_{mn}})x_2 + & \cdots & + & (A_{2(n-1)} - \frac{A_{2n}A_{m(n-1)}}{A_{mn}})x_{n-1} & = & 0 \\ \vdots & \vdots & \cdots & & \vdots & & \\ (A_{(m-1)1} - \frac{A_{(m-1)n}A_{m1}}{A_{mn}})x_1 + & (A_{(m-1)2} - \frac{A_{(m-1)n}A_{m2}}{A_{mn}})x_2 + & \cdots & + & (A_{(m-1)(n-1)} - \frac{A_{(m-1)n}A_{m(n-1)}}{A_{mn}})x_{n-1} & = & 0 \end{array}$$

(3) By induction, there is a nonzero solution $(x_1, \ldots, x_{n-1})$ to the $m - 1$ modified equations. Show that there is an x_n so that $(x_1, x_2, \ldots, x_{n-1}, x_n)$ is a solution to the original equations.