Problem Set 7 – due Monday August 2.

Algebraic numbers form a field

All these problems are related to the Linear Algebra problems from Problem Set 6.

Problem 1. (1) Write the following numbers in the form $a\sqrt[3]{4}+b\sqrt[3]{2}+c$, with a, b and c in $\mathbb{Q}$.

$$1,\ (\sqrt[3]{4}+\sqrt[3]{2}+1),\ (\sqrt[3]{4}+\sqrt[3]{2}+1)^2,\ (\sqrt[3]{4}+\sqrt[3]{2}+1)^3.$$

(2) Find a nonzero cubic polynomial in $\mathbb{Q}[x]$ which has $\sqrt[3]{4}+\sqrt[3]{2}+1$ as a root.

Problem 2. Let K be a field, let L be a larger field, and let α and β be elements of L which are algebraic over K of degrees a and b. Let $K[\alpha,\beta]$ be the set of elements in L which can be written in the form $\sum_{i,j} c_{ij}\alpha^i\beta^j$ with c_{ij} in K.

(1) Show that every element of $K[\alpha,\beta]$ can be written in the form $\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} c_{ij}\alpha^i\beta^j$ with c_{ij} in K. Note that we are **not** asking for this expression to be unique.
(2) Show that every element of $K[\alpha,\beta]$ is algebraic over K. In particular, show that $\alpha+\beta$ and $\alpha\beta$ are algebraic over K.

Problem 3. Let K be a field, let L be a larger field, and let K^{alg} be the subset of numbers in L which are algebraic over K. Show that K^{alg} is a field.

Derivatives

Let k be a field and let $f(x) = f_n x^n + f_{n-1}x^{n-1} + \cdots + f_2x^2 + f_1x + f_0$ be a polynomial with coefficients in k. We define the ***derivative of*** f, written $f'(x)$ to be the polynomial $nf_nx^{n-1} + (n-1)f_{n-1}x^{n-2} + \cdots + 2f_2x + f_1$. **In all the problems in this section, you should use this definition of derivative, and not any reference to limits.**

Problem 4. For any polynomials $f(x)$ and $g(x)$ in $k[x]$, prove:

(1) (**The sum rule**) $(f+g)'(x) = f'(x) + g'(x)$.
(2) (**The product rule**) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
(3) (**The chain rule**) $(f\circ g)'(x) = f'(g(x))g'(x)$.

Problem 5. Let $f(x)$ be a nonzero polynomial with coefficient in k, and let r be in k. We define r to be a ***root of multiplicity*** m if $f(x) = (x-r)^m g(x)$ with $g(r) \neq 0$. (So the multiplicity is zero if $f(r) \neq 0$.)

(1) Show that, if r is a root of $f(x)$ with multiplicity $m > 0$ then r has multiplicity $\geq m-1$ as a root of $f'(x)$.
(2) With notation as above, show that, if $m \neq 0$ in the field k, then r has multiplicity exactly $m-1$ as a root of $f'(x)$.
(3) Give an example of a $f(x)$ polynomial in $\mathbb{Z}_p[x]$ with a zero r of multiplicity p such that r has multiplicity $> p-1$ as a root of $f'(x)$.

Problem 6. This problem continues the ideas from the previous problem. Let k be a field and let $f(x)$ be a polynomial in $k[x]$. We define $f(x)$ to be ***separable*** if and only if $f(x)$ and $f'(x)$ are relatively prime

(1) Show that, if f is separable, then all the roots of f have multiplicity 1.
(2) Let L be a splitting field for $f(x)$, where $f(x) = c\prod(x-\theta_i)$. Show that $f(x)$ is separable if and only if the θ_i are distinct.
(3) Suppose that, in the field k, we do **not** have $m = 0$ for any positive integer m. Show that every irreducible polynomial in $k[x]$ is separable.