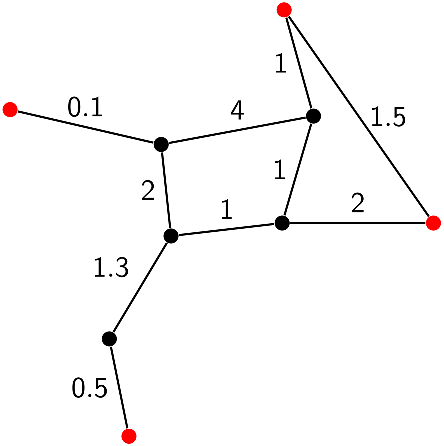
Combinatorics of electrical networks

Thomas Lam

September 2016

Electrical networks

An electrical resistor network is an undirected weighted graph Γ.



Edge weight $=$ conductance $= 1/\text{resistance}$
Some vertices are designated as boundary vertices. The rest are interior vertices.

Response matrix

The electrical properties are described by the response matrix

$$\Lambda(\Gamma) : \mathbb{R}^{\#\text{boundary vertices}} \longrightarrow \mathbb{R}^{\#\text{boundary vertices}}$$
$$\text{voltage vector} \longmapsto \text{current vector}$$

which gives the current that flows through the boundary vertices when specified voltages are applied.

Response matrix

The electrical properties are described by the response matrix

$$\Lambda(\Gamma) : \mathbb{R}^{\#\text{boundary vertices}} \longrightarrow \mathbb{R}^{\#\text{boundary vertices}}$$
$$\text{voltage vector} \longmapsto \text{current vector}$$

which gives the current that flows through the boundary vertices when specified voltages are applied.

$$\Lambda_{ij} = \text{current flowing through vertex } j \text{ when the voltage is set to 1 at vertex } i \text{ and 0 at all other vertices.}$$

Possibly surprisingly, $\Lambda(\Gamma)$ is a symmetric matrix.
If all vertices are considered boundary vertices, then $\Lambda(\Gamma)$ is simply the Laplacian matrix of Γ.

Axioms of electricity

The matrix $\Lambda(\Gamma)$ can be computed using only two axioms.

Kirchhoff's Law (1845)

The sum of currents flowing into an interior vertex is equal to 0.

Axioms of electricity

The matrix $\Lambda(\Gamma)$ can be computed using only two axioms.

Kirchhoff's Law (1845)

The sum of currents flowing into an interior vertex is equal to 0.

Ohm's Law (1827)

For each resistor we have

$$(V_1 - V_2) = I \times R$$

where
I = current flowing through the resistor
V_1, V_2 = voltages at the two ends of resistor
R = resistance of the resistor

To compute $\Lambda(\Gamma)$, we give variables to each edge (current through that edge) and each vertex (voltage at that vertex). Then solve a large system of linear equations.

Some basic problems

Inverse problem

Can we recover Γ from $\Lambda(\Gamma)$?

Applications of this to e.g. electrical impedance tomography (medical imaging technique).

Some basic problems

Inverse problem

Can we recover Γ from $\Lambda(\Gamma)$?

Applications of this to e.g. electrical impedance tomography (medical imaging technique).

Detection problem

Given a matrix M, how can we tell if $M = \Lambda(\Gamma)$ for some Γ?

Some basic problems

Inverse problem

Can we recover Γ from $\Lambda(\Gamma)$?

Applications of this to e.g. electrical impedance tomography (medical imaging technique).

Detection problem

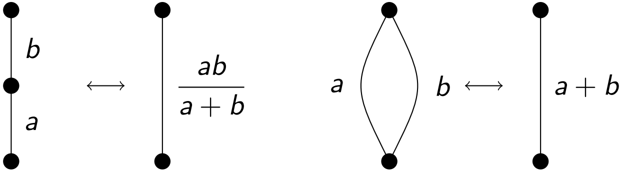
Given a matrix M, how can we tell if $M = \Lambda(\Gamma)$ for some Γ?

Equivalence problem

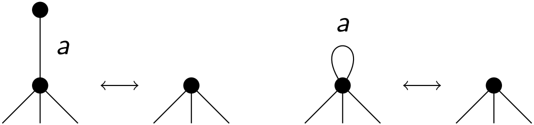
When do two networks Γ and Γ' satisfy $\Lambda(\Gamma) = \Lambda(\Gamma')$?

Electrically equivalent networks

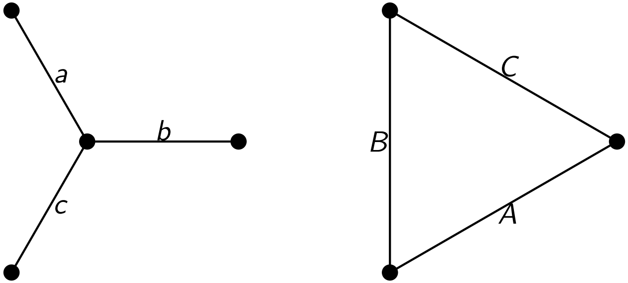
Series-parallel transformations:



Degenerate reductions:



$Y - \Delta$, or star-triangle transformation (Kennelly 1899)

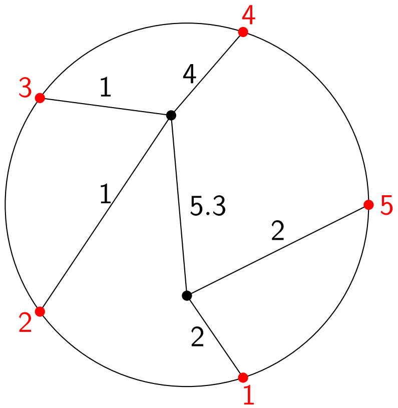


Γ Γ'

$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c},$$

$$a = \frac{AB + AC + BC}{A}, \quad b = \frac{AB + AC + BC}{B}, \quad c = \frac{AB + AC + BC}{C}.$$

Planar electrical networks



Planar electrical networks

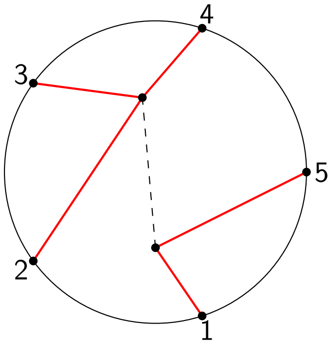
Theorem (Curtis-Ingerman-Morrow and Colin de Verdière-Gitler-Vertigan)

Consider planar electrical networks with n boundary vertices.

1. *Any two planar electrical networks* Γ, Γ' *such that* $\Lambda(\Gamma) = \Lambda(\Gamma')$ *are related by local electrical equivalences.*
2. *The space of response matrices consists of symmetric* $n \times n$ *matrices, with row sums equal to 0, and such that certain "circular minors" are nonnegative.*

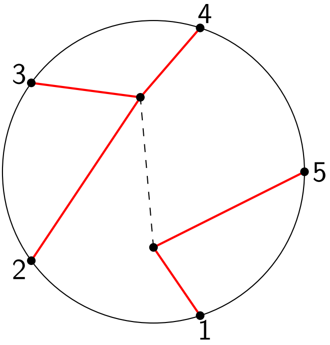
Groves (Carroll-Speyer and Kenyon-Wilson)

A grove F in Γ is a subforest such that every interior vertex is connected to some boundary vertex.



Boundary partitions

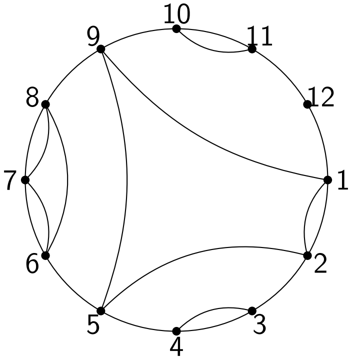
The boundary partition $\sigma(F)$ of a grove F is the noncrossing partition whose parts are boundary vertices belonging to the same component of F.



$$\sigma(F) = (2, 3, 4 \mid 1, 5)$$

Planarity $\implies$ noncrossing.

Noncrossing partitions



The noncrossing partition $\sigma = (1, 2, 5, 9 \mid 3, 4 \mid 6, 7, 8 \mid 10, 11 \mid 12)$.

Noncrossing partitions

Theorem

The number of noncrossing partitions on $[n]$ is equal to the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

For $n = 3$, we have 5 noncrossing partitions.

$$(123), (1|23), (12|3), (13|2), (1|2|3).$$

Grove measurements

Let $\mathcal{NC}_n$ denote the set of noncrossing partitions on $\{1, \ldots, n\}$.

Grove generating function

For $\sigma \in \mathcal{NC}_n$, and an electrical network Γ, define

$$L_\sigma(\Gamma) = \sum_{\sigma(F)=\sigma} \mathrm{wt}(F)$$

where the weight of a grove F is the product of the weights of the edges belonging to F.

Grove measurements

Let $\mathcal{NC}_n$ denote the set of noncrossing partitions on $\{1, \ldots, n\}$.

Grove generating function

For $\sigma \in \mathcal{NC}_n$, and an electrical network Γ, define

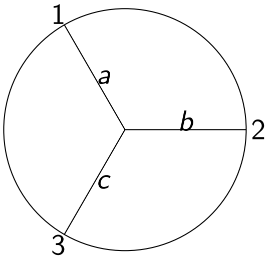
$$L_\sigma(\Gamma) = \sum_{\sigma(F)=\sigma} \mathrm{wt}(F)$$

where the weight of a grove F is the product of the weights of the edges belonging to F.

We collect all the L_σ's together to obtain a map

$$\Gamma \longmapsto \mathcal{L}(\Gamma) = (L_\sigma(\Gamma))_{\sigma \in \mathcal{NC}_n} \in \mathbb{P}^{\mathcal{NC}_n}.$$

Example



Γ

$$
\begin{aligned}
L_{1|2|3} &= a + b + c, \\
L_{12|3} &= ab, \\
L_{1|23} &= bc, \\
L_{13|2} &= ac, \\
L_{123} &= abc \\
\mathcal{L}(\Gamma) &= (a + b + c : ab : bc : ac : abc) \in \mathbb{P}^4
\end{aligned}
$$

Capturing electrical equivalence

Two networks are electrically equivalent if they have the same response matrix.

Theorem

Γ and Γ' are electrically equivalent if and only if $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$.

Counting forests captures electrical properties.

Capturing electrical equivalence

Two networks are electrically equivalent if they have the same response matrix.

Theorem

Γ and Γ' are electrically equivalent if and only if $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$.

Counting forests captures electrical properties.

If part follows from the formula essentially due to Kirchhoff:

$$\Lambda_{ij} = \frac{L_{(i,j|\text{singletons})}}{L_{(\text{all singletons})}}$$

Matrix-tree theorem

A much more famous formula of Kirchhoff (no planarity is needed):

Theorem

We have

$$\tilde{\det}(\Lambda) = \frac{L_{(\textit{all connected})}}{L_{(\textit{all singletons})}}$$

where $\tilde{\det}$ *denotes the reduced determinant: remove one row and one column before taking the determinant.*

The generating function $L_{(\text{all connected})}$ counts spanning trees of Γ.

Matrix-tree theorem

A much more famous formula of Kirchhoff (no planarity is needed):

Theorem

We have

$$\tilde{\det}(\Lambda) = \frac{L_{(\textit{all connected})}}{L_{(\textit{all singletons})}}$$

where $\tilde{\det}$ *denotes the reduced determinant: remove one row and one column before taking the determinant.*

The generating function $L_{(\text{all connected})}$ counts spanning trees of Γ.

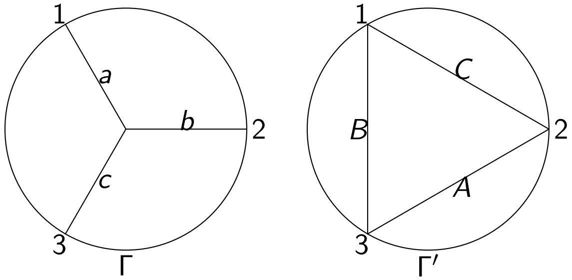
Kenyon and Wilson give (alternating) formulae for all minors of Λ.

Invariance under electrical equivalences

Kirchhoff's formula allows one to recover $\Lambda(\Gamma)$ (response matrix) from $\mathcal{L}(\Gamma)$ (grove counts).

To show that $\mathcal{L}(\Gamma)$ depends only on $\Lambda(\Gamma)$, one checks that $\mathcal{L}(\Gamma) \in \mathbb{P}^{\mathcal{NC}_n}$ is invariant under series-parallel transformations, degenerate reductions, and star-triangle transformations.

$Y - \Delta$-transformation



$$\begin{pmatrix} \mathcal{L}(\Gamma) \\ \mathcal{L}(\Gamma') \end{pmatrix} = \begin{pmatrix} a+b+c & ab & bc & ac & abc \\ 1 & C & A & B & AB+BC+AB \end{pmatrix}$$

The same point in $\mathbb{P}^4$ under

$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c}.$$

Compactification

Let us use nonnegative edge weights. The image of the map $\Gamma \mapsto \mathcal{L}(\Gamma)$ is not compact. We let

$$E_n = \overline{\{\mathcal{L}(\Gamma) \mid \Gamma \text{ planar electrical network } \}} \subset \mathbb{P}^{\mathcal{NC}_n}$$

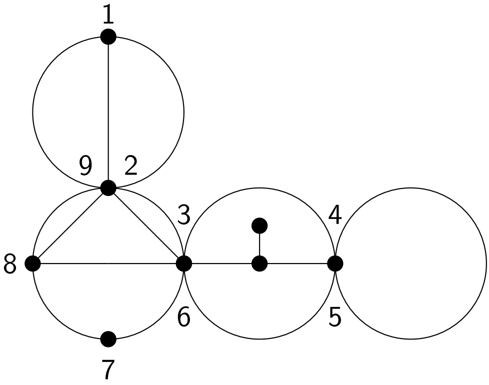
denote the closure of the image, called the

compactified space of planar electrical networks.

The topology on this space corresponds to continuously varying conductances. When a conductance goes to 0 or ∞, we change the combinatorics.

Cactus networks

Roughly speaking, a point $\mathcal{L} \in E_n$ is represented by an electrical network where some of the boundary points have been glued together (or "shorted"), in a planar way.



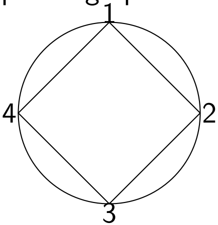
(Conductances are not shown.)

Electroids

The electroid $\mathcal{E}(\Gamma)$ of $\Gamma \in E_n$ is the set

$$\mathcal{E}(\Gamma) = \{\sigma \mid L_\sigma(\Gamma) \neq 0\} \subset \mathcal{NC}_n.$$

These are noncrossing partitions for which there exist groves inducing such a partition.
This is an invariant of a planar graph.



$$\mathcal{E} = \{(1|2|3|4), (12|3|4), (23|1|4), (34|1|2), (14|23), (12|34), (14|23), (123|4), (234|1), (134|2), (124|3), (1234)\}$$

missing: $(13|2|4)$ and $(24|1|3)$

Electroid stratification

Question

What are all possible electroids? How many electroids are there?

We have the electroid stratification

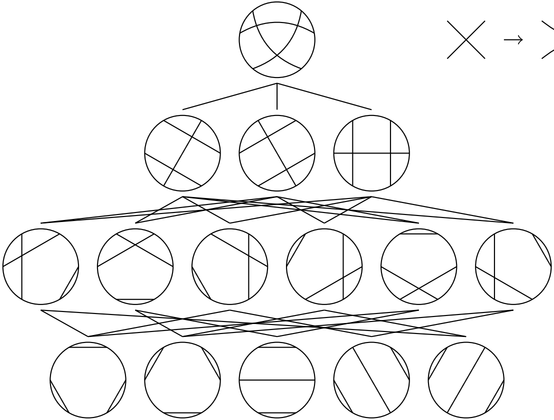
$$E_n = \bigsqcup_{\mathcal{E}} E_{\mathcal{E}}.$$

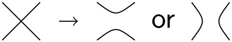
This stratification is analogous to the matroid stratification of a Grassmannian. More precisely, it is an analogue of the positroid (positive matroid) stratification.

Noncrossing partitions $\leftrightarrow$ Bases of matroids

Uncrossing poset for matchings

The set P_n of matchings on $\{1, 2, \ldots, 2n\}$ is a graded poset with rank function $c(\tau) =$ number of crossings. (Studied by Alman-Lian-Tran, Kenyon, Huang-Wen-Zie, Kim-Lee, L., ...)



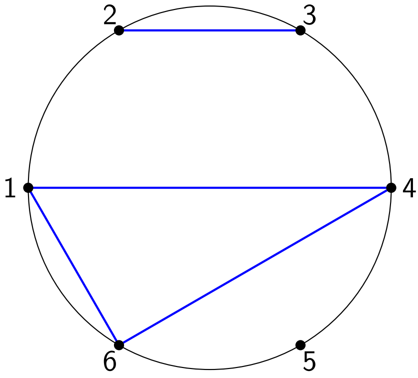


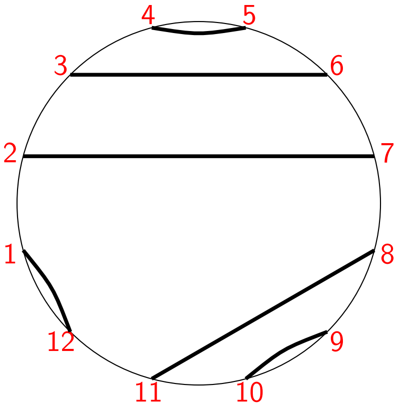
$|P_n| = (2n-1).(2n-3).\ldots.5.3.1 \qquad |P_3| = 15$

Noncrossing partitions to noncrossing matchings

Bijection:

$$\sigma \in \mathcal{NC}_n \longmapsto \tau(\sigma) \in P_n$$





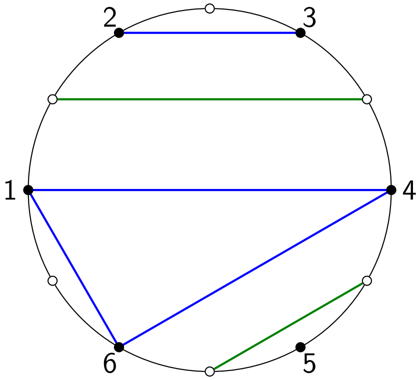
$$\sigma = (146|23|5)$$

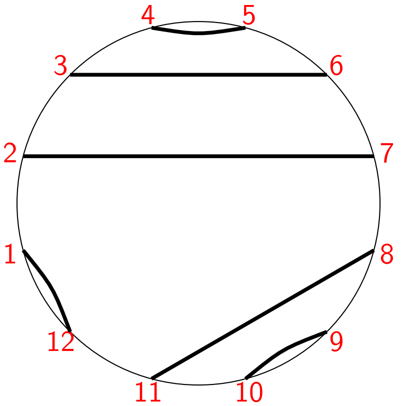
$$\tau(\sigma) = \{(1,12),(2,7),(3,6),(4,5),(8,11),(9,10)\}$$

Noncrossing partitions to noncrossing matchings

Bijection:

$$\sigma \in \mathcal{NC}_n \longmapsto \tau(\sigma) \in P_n$$





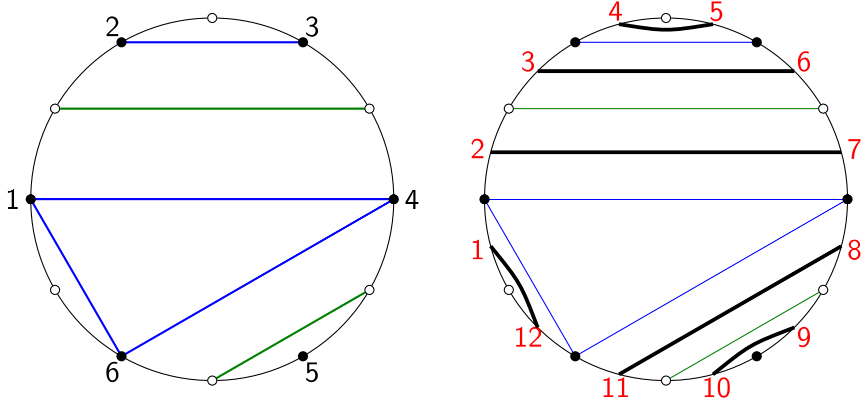
$$\sigma = (146|23|5)$$

$$\tau(\sigma) = \{(1,12),(2,7),(3,6),(4,5),(8,11),(9,10)\}$$

Noncrossing partitions to noncrossing matchings

Bijection:

$$\sigma \in \mathcal{NC}_n \longmapsto \tau(\sigma) \in P_n$$



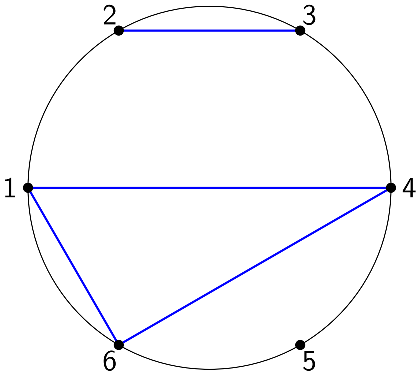
$$\sigma = (146|23|5)$$

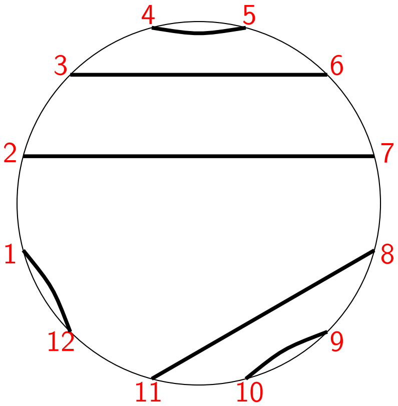
$$\tau(\sigma) = \{(1,12),(2,7),(3,6),(4,5),(8,11),(9,10)\}$$

Noncrossing partitions to noncrossing matchings

Bijection:

$$\sigma \in \mathcal{NC}_n \longmapsto \tau(\sigma) \in P_n$$





$$\sigma = (146|23|5)$$

$$\tau(\sigma) = \{(1,12),(2,7),(3,6),(4,5),(8,11),(9,10)\}$$

Matchings classify electroid strata

Theorem (L.)

There is a bijection $\tau \leftrightarrow \mathcal{E}(\tau)$ between matchings and electroids, given by

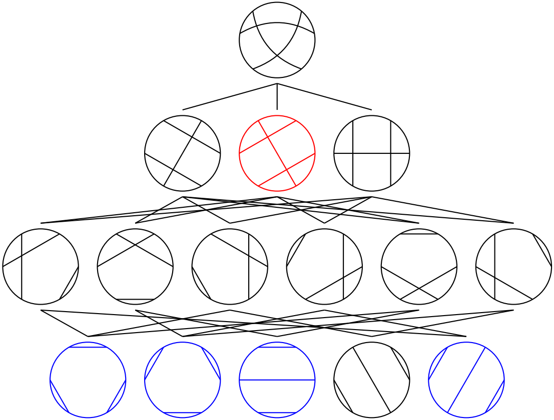
$$\mathcal{E}(\tau) = \{\sigma | \tau(\sigma) \leq \tau\} \subseteq \mathcal{NC}_n.$$

Matchings classify electroid strata

Theorem (L.)

There is a bijection $\tau \leftrightarrow \mathcal{E}(\tau)$ *between matchings and electroids, given by*

$$\mathcal{E}(\tau) = \{\sigma | \tau(\sigma) \leq \tau\} \subseteq \mathcal{NC}_n.$$



$$|\mathcal{E}(\tau)| = 4$$

Matchings are in bijection with electroids

Recall that we defined a stratification

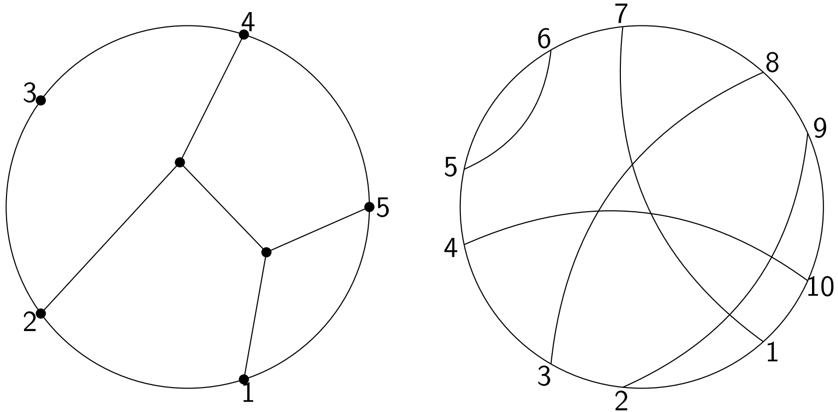
$$E_n = \bigsqcup_{\mathcal{E}} E_{\mathcal{E}}$$

Theorem (L.)

Label $E_{\mathcal{E}(\tau)}$ by E_τ.

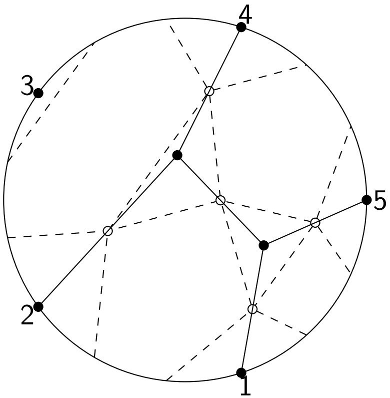
1. *For any $\tau \in P_n$, we have $E_\tau \simeq \mathbb{R}_{>0}^{c(\tau)}$*
2. *For any $\tau \in P_n$, we have $\overline{E_\tau} = \bigsqcup_{\tau' \leq \tau} E_{\tau'}$.*

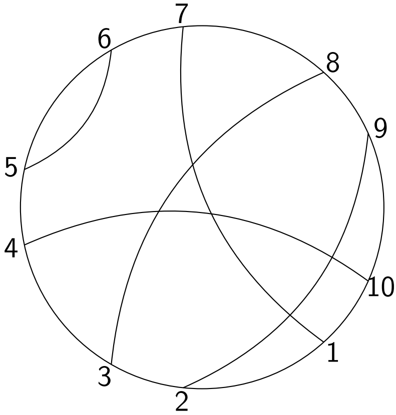
Medial graph



Electrical network $\longrightarrow$ Medial graph $\longrightarrow$ Matching on $[2n]$

Medial graph





Electrical network ⟶ Medial graph ⟶ Matching on $[2n]$

Some open problems

Question

Let $\tilde{E}_n$ be the Zariski closure of E_n in $\mathbb{P}^{\mathcal{NC}_n}(\mathbb{C})$. What are the dimensions of $\Gamma(\tilde{E}_n, \mathcal{O}(d))$?

For $d = 1$, it is the Catalan number C_n.

Question

Is E_n homeomorphic to a ball? Is P_n shellable?

We know P_n is graded and Eulerian, and we know E_n is contractible.

Some open problems

Problem

Generalize the theory to electrical networks embedded on a surface.

Some work has been done on the cylinder and torus.

Question

What is the analogue of the matroid axioms for electroids?

$$\text{Noncrossing partitions} \longleftrightarrow \text{Bases}$$
$$\text{Partial noncrossing partitions} \longleftrightarrow \text{Independent sets, circuits}$$
$$\vdots$$

Thank you!