On the topology of totally positive spaces

Thomas Lam, University of Michigan
joint work with Pavel Galashin and Steven Karp (thanks for slides!)

October 20th, 2018
Ann Arbor

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

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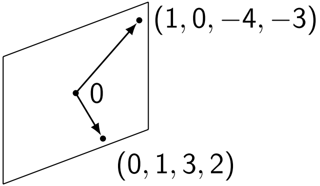
$V :=$  $\in \mathrm{Gr}_{2,4}$

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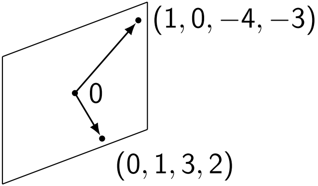

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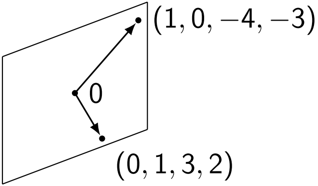

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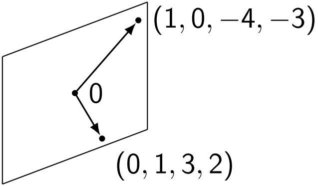

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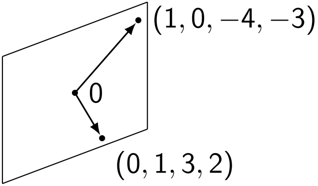

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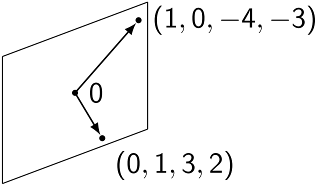
$V :=$ 

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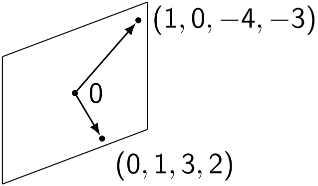

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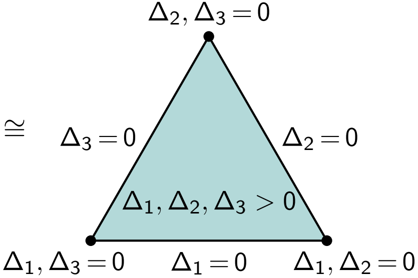

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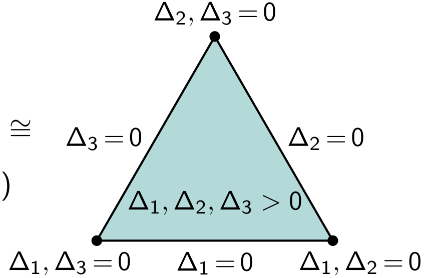

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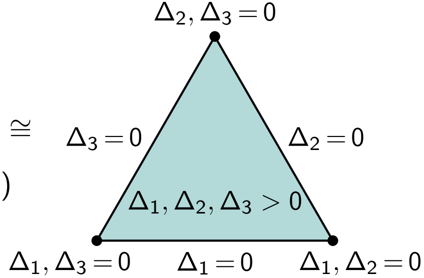

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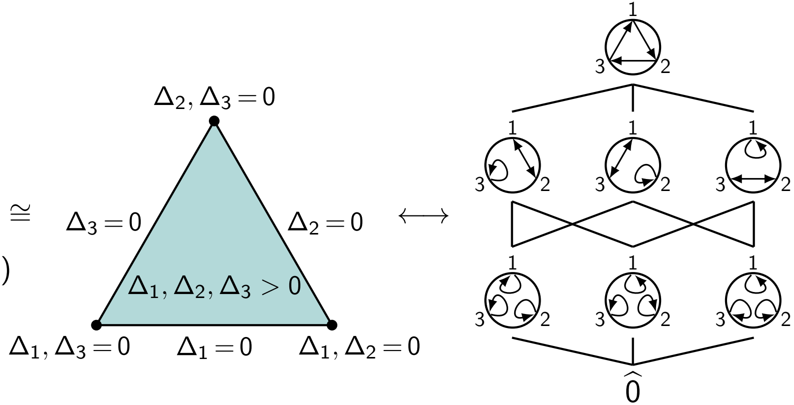

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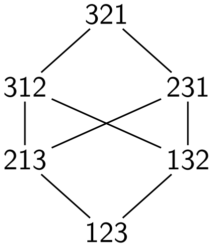

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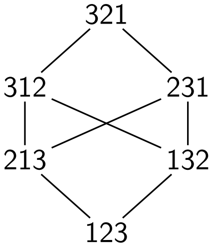

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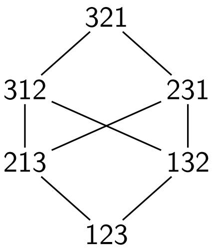

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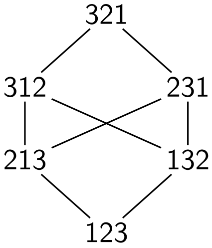

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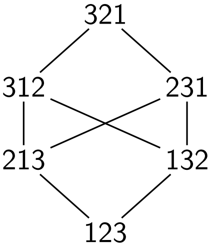

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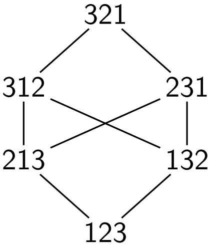

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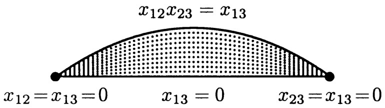

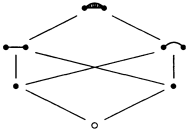

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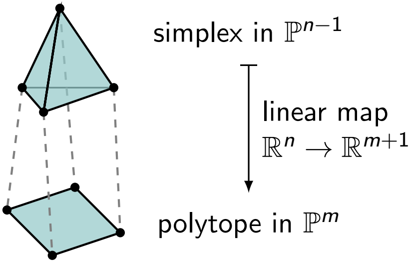

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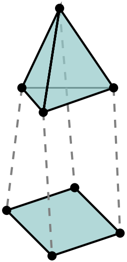

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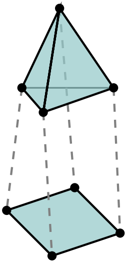

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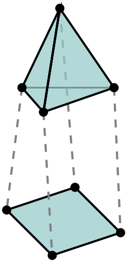

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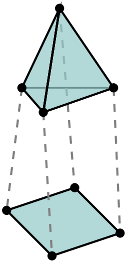

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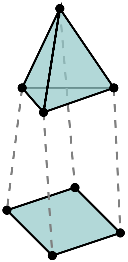

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- When Z has positive maximal minors, the Grassmann polytope is called an *amplituhedron*. Amplituhedra generalize cyclic polytopes into the Grassmannian. They were introduced by Arkani-Hamed and Trnka (2014).

Old idea

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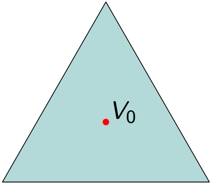

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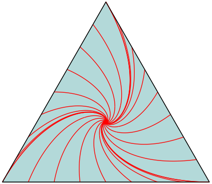

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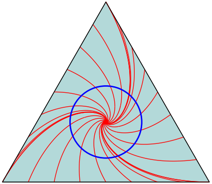

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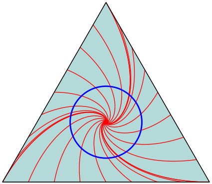
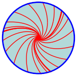


$\mapsto$



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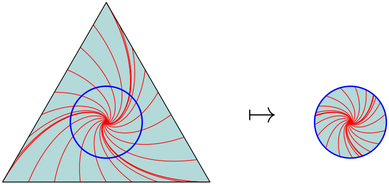

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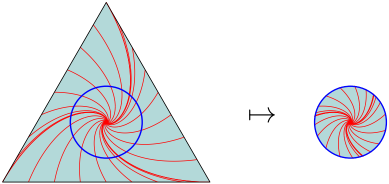

- Unfortunately, there exists closed positroid cells $\Pi_f^{\geq 0}$ such that every smooth vector field vanishes at a particular point $X_{\mathrm{sing}} \in \Pi_f^{\geq 0}$.

Use Poincaré conjecture

Theorem (Smale, Freedman, Perelman)

Let C be a compact contractible topological manifold with boundary, such that its boundary ∂C is homeomorphic to a sphere of dimension $\dim(C) - 1$*. Then C is homeomorphic to a closed ball.*

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- If both $\Pi_f^{>0}$ and $\mathrm{Link}(f,g)$ are known to be balls, then this shows that $\Pi_g^{\geq 0}$ is a manifold with boundary along $\Pi_f^{>0} \subset \partial \Pi_g^{\geq 0}$.

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Thank you!