Electrical networks and Lie theory

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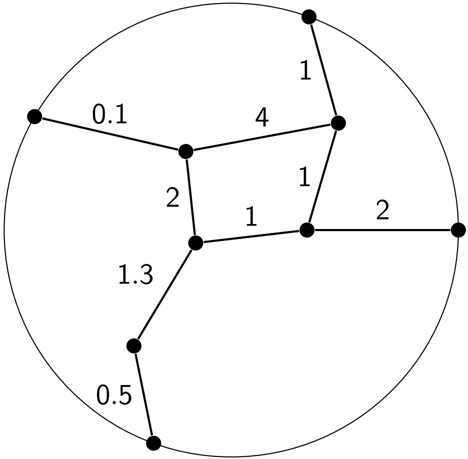
Electrical networks

An electrical network consisting only of resistors can be modeled by an undirected weighted graph Γ.



Edge weight = conductance = 1/resistance

Response matrices

The electrical properties are described by the response matrix

$$L(N) : \mathbb{R}^{\#\text{boundary vertices}} \to \mathbb{R}^{\#\text{boundary vertices}}$$

$$\text{voltage vector} \longmapsto \text{current vector}$$

which gives the current that flows through the boundary vertices when specified voltages are applied.

Axioms of electricity

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Ohm's Law

For each resistor we have

$$(V_1 - V_2) = I \times R$$

where
$I =$ current flowing throught the resistor
$V_1, V_2 =$ voltages at two ends of resistor
$R =$ resistance of the resistor

To compute $L(N)$, we give variables to each edge (current through that edge) and each vertex (voltage at that vertex). Then solve a large system of linear equations.

Some basic problems

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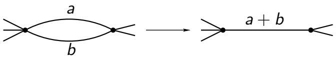
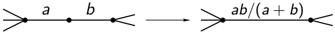
Equivalence problem

When do two networks N and N' satisfy $L(N) = L(N')$?

Electrical relations

Series-parallel transformations:

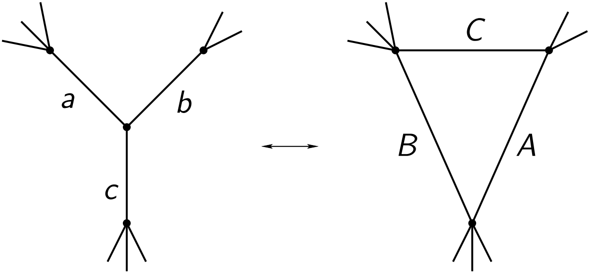




Electrical relations

Series-parallel transformations:

$Y - \Delta$, or star-triangle transformation:



$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c},$$

$$a = \frac{AB+AC+BC}{A}, \quad b = \frac{AB+AC+BC}{B}, \quad c = \frac{AB+AC+BC}{C}.$$

Planar electrical networks

Theorem (Curtis-Ingerman-Morrow and Colin de Verdière-Gitler-Vertigan)

Consider planar electrical networks with n boundary vertices.

1. *Any two planar electrical networks N, N' such that $L(N) = L(N')$ are related by local electrical equivalences.*
2. *The space P_n of response matrices consists of symmetric $n \times n$ matrices, with row sums equal to 0, and such that certain "circular minors" are nonnegative.*
3. *We have*

$$P_n = \sqcup_{G_i} \mathbb{R}_{>0}^{\#\text{edges in } G_i}$$

where $\{G_i\}$ is a finite collection of unlabeled graphs, and each cell is parametrized by giving arbitrary positive weights to the edges.

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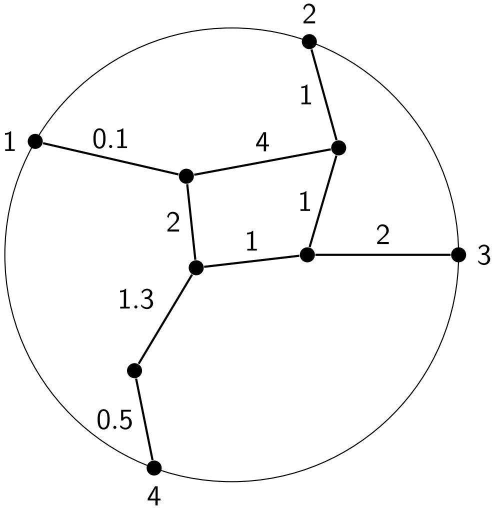
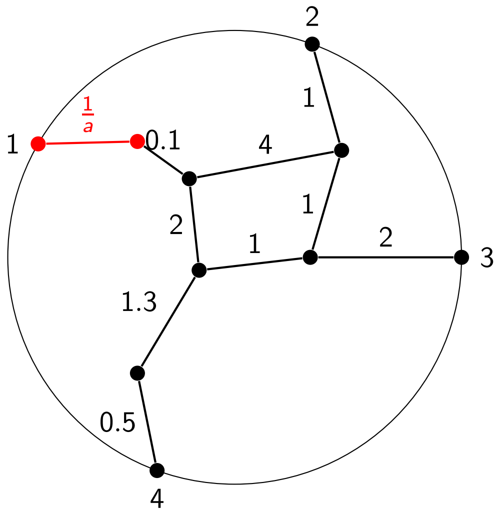
Recently, Kenyon-Wilson developed a theory connecting electrical networks to the enumeration of "groves" in graphs.

Operations on planar electrical networks

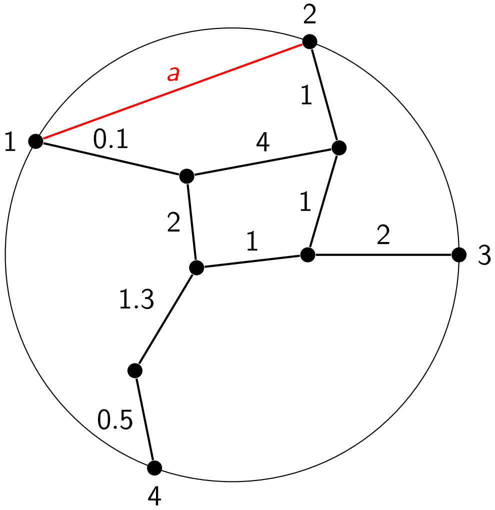
$N =$ 

Operations on planar electrical networks

$u_1(a).N =$ 

Operations on planar electrical networks

$u_2(a).N =$



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The operations $u_i(a)$ satisfy the relations:

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$$u_i(a)u_j(b)u_i(c) = u_j(\frac{bc}{a+c+abc})u_i(a+c+abc)u_j(\frac{ab}{a+c+abc})$$

due to $Y - \Delta$ equivalence.

Lusztig's braid relation in total positivity

$$x_1(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad x_2(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies

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Very similar to

$$u_i(a)u_j(b)u_i(c) = u_j(\frac{bc}{a+c+abc})u_i(a+c+abc)u_j(\frac{ab}{a+c+abc})$$

Electrical Lie algebras $\mathfrak{el}_{2n}$

generators: $e_1, e_2, \ldots, e_{2n}$
relations:

- $[e_i, e_j] = 0$ if $|i - j| \geq 2$
- $[e_i, [e_i, e_{i\pm1}]] = -2e_i$

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Theorem (L.-Pylyavskyy)

1. *We have* $\mathfrak{el}_{2n} = \mathfrak{sp}_{2n}$ *and so* $\dim(\mathfrak{el}_{2n}) = \dim(n_+(\mathfrak{sl}_{2n+1}))$.
2. *In the simply-connected Lie group* $EL_{2n}(\mathbb{R})$ *the elements*

$$u_i(t) = \exp(t\, e_i)$$

satisfy all the relations from before.

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There appear to electrical Lie algebras associated to any Dynkin diagram of a finite-dimensional simple Lie algebra. Yi Su has described the type C analogue and associated action on mirror-symmetric planar electrical networks.