The shape of a random affine Weyl group element

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Infinite reduced words

Coxeter groups

A Coxeter group (W, S) is a group generated by a set $S = \{s_1, s_2, \dots, s_r\}$ of simple generators which are involutions satisfying relations of the form

$$(s_i s_j)^{m_{ij}} = 1$$

Definition

A word $i_1 i_2 \cdots i_\ell$ is a reduced word if ℓ is minimal amongst expressions $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for w.

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An infinite reduced word is a sequence $i_1 i_2 i_3 \cdots$ such that each initial subsequence $i_1 i_2 \cdots i_k$ is reduced.

Example (Symmetric group S_3)

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Example (Affine symmetric group $\tilde{S}_3$)

$\tilde{S}_3$ is generated by involutions s_0, s_1, s_2 with relations

$$s_1 s_2 s_1 = s_2 s_1 s_2 \qquad s_0 s_1 s_0 = s_1 s_0 s_1 \qquad s_2 s_0 s_2 = s_0 s_2 s_0$$

$012012012012\cdots$ is an infinite reduced word

Basic question

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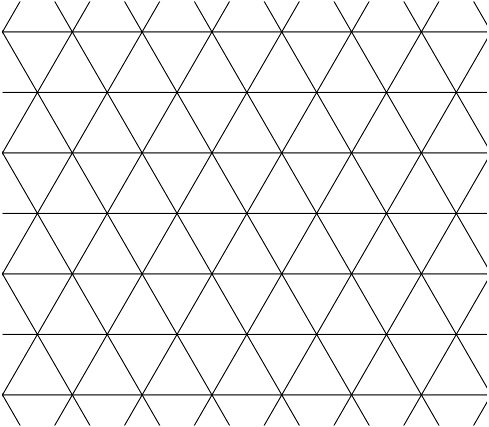
1. Classification (up to braid equivalence), and the limit weak order for infinite reduced words in affine Weyl groups (studied with P. Pylyavskyy, also work of Ito, Cellini-Papi).
2. Infinite reduced words as geodesics in Coxeter (and Davis) complexes and relation to Tits metric on the visual boundary (studied with A. Thomas).

Affine Coxeter groups

We will restrict ourselves to the case that W is an affine Weyl group.

An affine Weyl group is a group generated by affine reflections acting cocompactly on a Euclidean space.

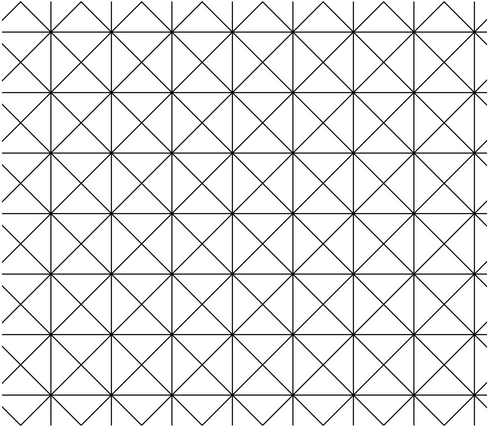
The $\tilde{A}_2$ arrangement



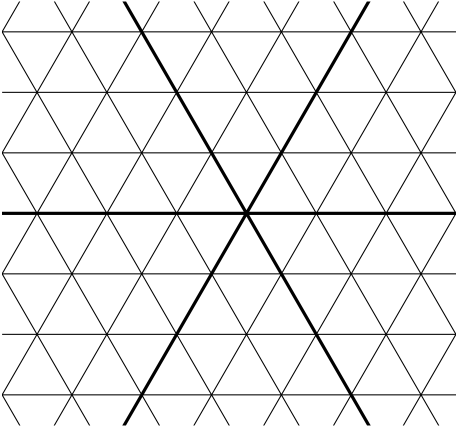
The affine symmetric group $\tilde{S}_3$ acts simply-transitively on the alcoves of this arrangement.

$\widetilde{A_1 \times A_1}$ arrangement

$\tilde{B}_2$ arrangement

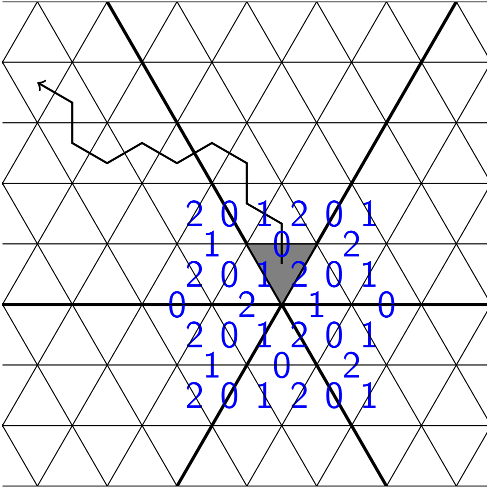


Weyl chambers



The Weyl chambers are formed by the hyperplanes passing through the origin. Here there are six Weyl chambers, in bijection with the finite Weyl group S_3 (generated by reflections in these three hyperplanes).

Infinite reduced word = walk in affine Coxeter arrangement



The above walk corresponds to the infinite reduced word $0120210201\cdots$.

REDUCED = no hyperplane crossed more than once

Reduced random walk

Fix an affine Weyl group W.

The reduced random walk $X = (X_0, X_1, \ldots)$ is a sequence of alcoves in the affine Coxeter arrangement of W, where each step is chosen uniformly at random amongst choices which keep the walk reduced.

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Easy Facts:

1. These walks can never "get stuck".
2. This process is a transient Markov chain.

Main Theorem 1

Fix an affine Weyl group W. Let $X = (X_0, X_1, \ldots)$ be the reduced random walk.

Theorem (L.)

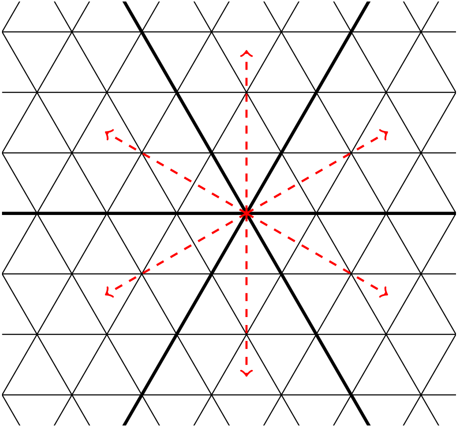
There exists a unit vector $\psi \in V$ such that almost surely

$$\lim_{k \to \infty} \nu(X_k) \in W \cdot \psi$$

where $\nu(X_k)$ denotes the unit vector pointing towards the center of the alcove X_k.

In other words, there is a finite collection $\{W \cdot \psi\}$ such that with probability one, the reduced random walk asymptotically approaches one of these directions.

Asymptotic directions



The asymptotic directions for $\tilde{S}_3$.

A Markov chain on W_{fin}

Define a Markov chain on the finite Weyl group W_{fin} with transitions of probability $1/r$ (with $r = \dim V + 1$) given by either

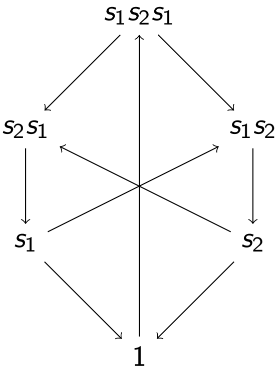
$$w \to s_i w \qquad \text{if } \ell(s_i w) < \ell(w)$$

or

$$w \to r_\theta w \qquad \text{if } \ell(r_\theta w) > \ell(w)$$

Here r_θ is the longest reflection in W_{fin}, and extra transitions from w to w are added to make this a Markov chain.

The Markov chain for S_3



All transitions have probability $1/3$. Add self-loops to make this a Markov chain.

Main Theorem 2

Theorem (L.)

The Markov chain on W_{fin} has a unique stationary distribution $\zeta : W_{\mathrm{fin}} \to \mathbb{R}$. We have

$$\psi = \frac{1}{Z} \sum_{w \in W_{\mathrm{fin}} : \ell(r_\theta w) > \ell(w)} \zeta(w) w^{-1}(\theta^\vee).$$

Probabilities of staying in a Weyl chamber

Since there are only finitely many Weyl chambers, the reducedness condition implies that every reduced walk will eventually stay in some Weyl chamber C_w. Write

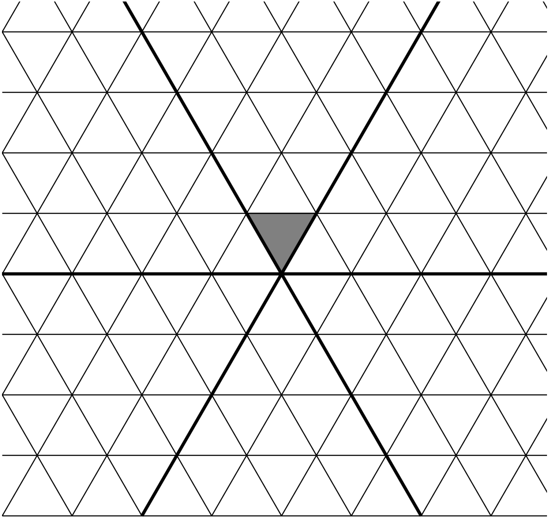
$$X \in C_w$$

for this event.

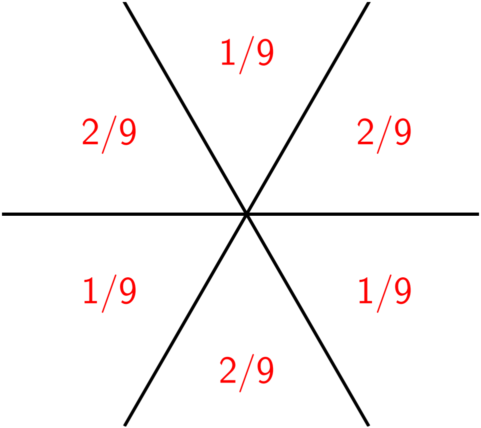
Question

What is $\mathrm{Prob}(X \in C_w)$?

Can you guess $\mathrm{Prob}(X \in C_w)$?



The answer



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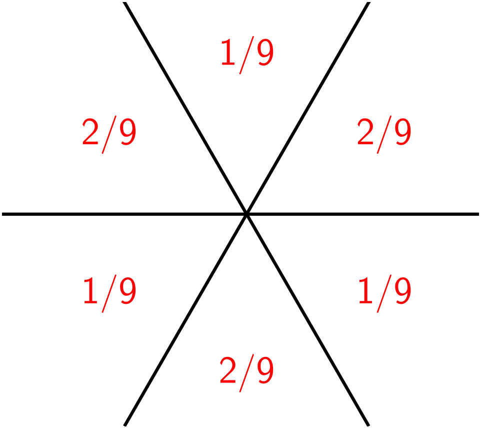

In four dimensions, one chamber is 96 times more likely than the least likely chamber.

Main Theorem 3

Theorem (L.)

$$\mathrm{Prob}(X \in C_w) = \zeta(w^{-1} w_0)$$

where $w_0 \in W_{\mathrm{fin}}$ *is the longest element of* W_{fin}.

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Conjecture

Let $W = \tilde{S}_n$.

1. ψ is in the same direction as $\rho^\vee$.
2. $$\frac{\mathrm{Prob}(X \in C_w)}{\mathrm{Prob}(X \in C_1)} \in \mathbb{Z}.$$

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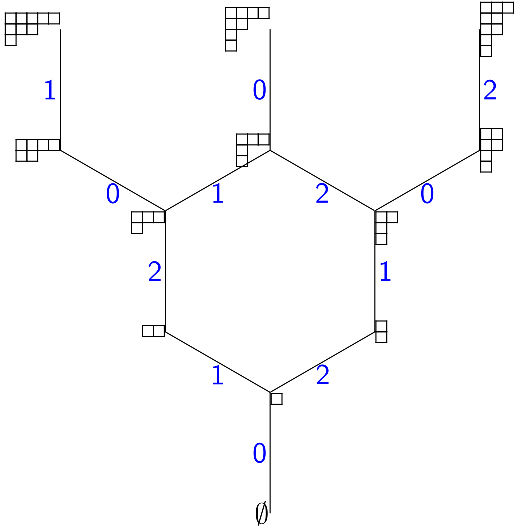
Many more conjectures for a multivariate generalization of the Markov chain on S_n with L. Williams, suggesting very interesting enumeration!

n-cores

n-cores are a special class of partitions. Here we illustrate the bijection between 3-cores and Grassmannian elements of $\tilde{S}_3$.



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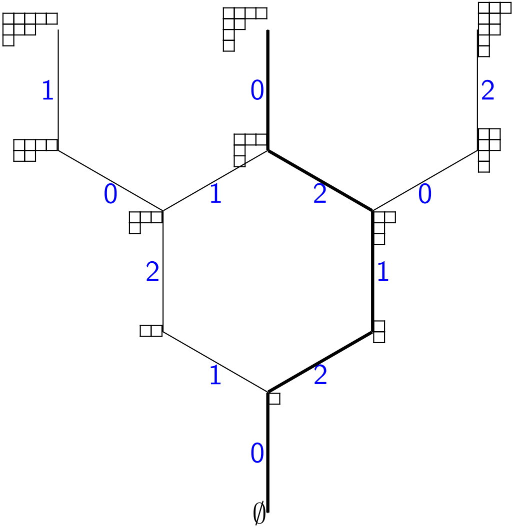

The reduced word $02120\cdots$ gives the thickened line.

The limiting shape of a random n-core.

Corollary

There exists a piecewise-linear curve C_n such that most large random n-cores (grown by the "reduced" process) has a shape arbitrarily close to C_n.

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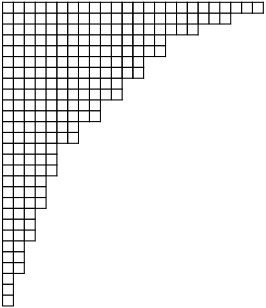

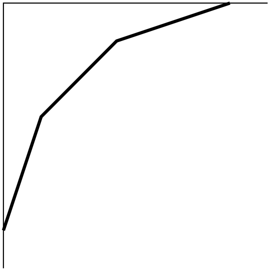

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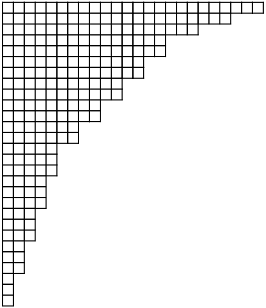
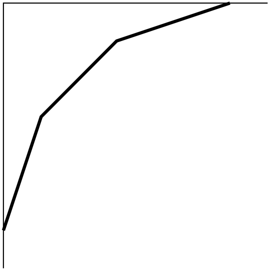




This might be compared with Kerov and Vershik's work on the shape of a random partition.

Taking $n \to \infty$

As $n \to \infty$, the piecewise-linear curve C_n, suitably scaled, approaches (one branch of) the continuous conic

$$\sqrt{x} + \sqrt{y} = 1$$

This curve has previously appeared as the limiting shape of another random process...

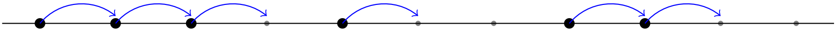
Continuous time TASEP

Continuous time TASEP on the integer lattice:

An independent random variable (waiting time) with exponential distribution is associated to each particle. The particle can jump only if the site immediately to the right is empty.

Continuous time TASEP

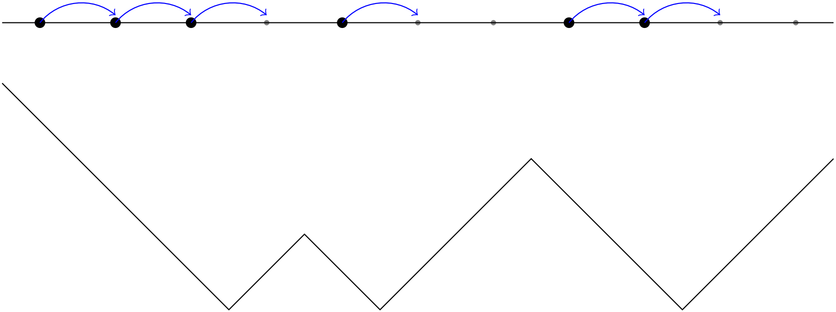
Continuous time TASEP on the integer lattice:



Initial configuration:

Continuous time TASEP

Continuous time TASEP on the integer lattice:



Each configuration is associated with a piece-wise linear curve, or Young diagram.

Continuous time TASEP

Continuous time TASEP on the integer lattice:

Johansson showed that the "limiting shape" of continuous time TASEP with exponential waiting time is exactly the same curve

$$\sqrt{x} + \sqrt{y} = 1$$

So for the affine symmetric group $W = \tilde{S}_n$, and conditioning our random walk to stay in the fundamental chamber, we obtain a periodic analogue of continuous time TASEP: particles separated by distance n are conditioned to jump simultaneously.