Total positivity in combinatorics and representation theory

Thomas Lam

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But

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 3 \end{pmatrix}$$

is TP.

Totally positive kernels

Examples of total nonnegative matrices first arose in analysis. The kernels $K(x, y) = e^{xy}$ and $K(x, y) = e^{-(x-y)^2}$ are totally nonnegative in the sense that the matrix

$$M = (K(x_i, y_j))_{i,j=1}^{k}$$

is totally nonnegative for every

$$x_1 < x_2 < \cdots < x_k$$
$$y_1 < y_2 < \cdots < y_k$$

We call these totally positive kernels.

Properties of totally positive matrices

Theorem (Gantmacher and Krein)

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Theorem (Schoenberg, Gantmacher and Krein, Karlin)

Let $K(x, y)$ be a TP kernel and $f(y)$ a function satisfying suitable integrability conditions. Then

$$g(x) = \int_{\mathbb{R}} K(x, y) f(y) dy$$

has no more sign-changes than $f(y)$.

Total positivity in $GL_n(\mathbb{R})$

Theorem (Loewner-Whitney)

$GL_n(\mathbb{R})_{\geq 0}$ is the semigroup generated by the positive diagonal matrices and positive Chevalley generators $e_i(t), f_i(t)$ with $t \geq 0$

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$$\mathrm{diag}(t_1, t_2, t_3, t_4) = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} \qquad t_1, t_2, t_3, t_4 > 0$$

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$$e_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad f_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Totally positive functions

A sequence $a_0, a_1, \cdots$ of real numbers is a totally positive sequence if the infinite Toeplitz matrix

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_0 & a_1 & a_2 & a_3 & \cdots \\ \cdots & 0 & a_0 & a_1 & a_2 & \cdots \\ \cdots & 0 & 0 & a_0 & a_1 & \cdots \\ \cdots & 0 & 0 & 0 & a_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is TNN (caution: not TP!).

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$$a_i \geq 0 \qquad a_1^2 - a_0 a_2 \geq 0 \qquad a_1^3 + a_3 a_0^2 - 2a_0 a_1 a_2 \geq 0....$$

The formal power series $a(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ is then called a totally positive function. Totally positive functions form a semigroup.

Edrei-Thoma theorem

Fact: If $a(t)$ is totally positive, then it is automatically a meromorphic function, holomorphic in a neighborhood of 0.

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Theorem

Every normalized ($a_0 = a_1 = 1$) totally positive sequence has generating function of the form

$$a(t) = e^{\gamma t} \prod_{i=1}^{\infty} \frac{1 + \alpha_i t}{1 - \beta_i t}$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq 0$, $\gamma \geq 0$, *and* $\gamma + \sum_i (\alpha_i + \beta_i) = 1$. *Conversely, all such sets of parameters give a normalized totally positive function.*

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The proof of this theorem relies on deep results in complex analysis (Nevanlinna theory).

Totally positive functions parametrize many things

Theorem (Thoma, Kerov-Vershik)

The following sets are in bijection (homeomorphic):

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- *(Normalized) homomorphisms* $\phi : \mathrm{Sym} \to \mathbb{R}$ *such that* $\phi(s_\lambda) \geq 0$ *for each Schur function* s_λ.

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- *(Normalized) homomorphisms $\phi : \mathrm{Sym} \to \mathbb{R}$ such that $\phi(s_\lambda) \geq 0$ for each Schur function s_λ.*
- *Extremal Markov chains on Young's lattice of partitions, such that the probability of a tableau only depends on its shape.*

The infinite symmetric group

Infinite symmetric group

The symmetric group S_n permuting n elements embeds into S_{n+1} as the subgroup fixing $n+1$. The inductive limit S_∞

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Under the correspondence (Thoma)

$$\{\text{totally positive functions}\} \leftrightarrow \{\text{characters of } S_\infty\}$$

one has (Okounkov)

$$\text{poles and zeroes } \{\alpha_i, \beta_i\} \leftrightarrow \text{atoms of particular spectral measure}$$

Partitions

A partition λ of n is a sequence

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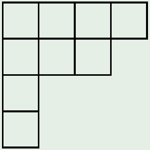
Example

$(4, 3, 1, 1)$ is a partition of 9.



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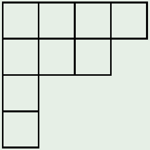

We have

$$\text{irreps of } S_n \stackrel{1:1}{\longleftrightarrow} \text{partitions of } n$$

Symmetric functions

Sym denotes the ring of formal power series of bounded degree in the variables $x_1, x_2, \ldots$, invariant under action of S_∞ on the indices. There is an isomorphism (Frobenius character)

$$\mathrm{Sym} = \mathrm{Sym}_{\mathbb{R}} \cong \oplus_{n \geq 0} \mathrm{Rep}(S_n) \otimes \mathbb{R}$$

where

$$\text{irrep labeled by } \lambda \leftrightarrow \text{Schur function } s_\lambda(x_1, x_2, \ldots).$$

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A character $\chi : S_\infty \to \mathbb{C}$ gives rise to a homomorphism $\phi : \mathrm{Sym} \to \mathbb{R}$, where

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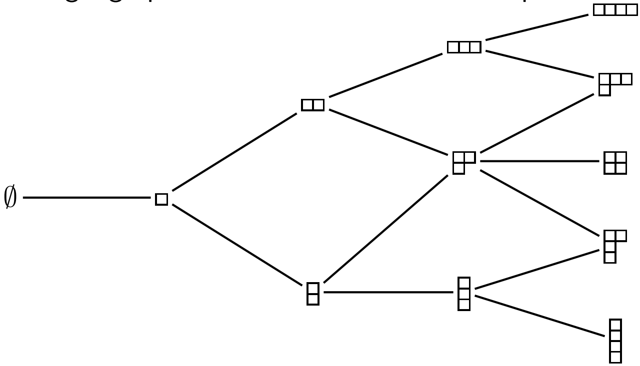
and

poles and zeroes $\{\alpha_i, \beta_i\} \leftrightarrow$ what to specialize the variables $x_1, x_2, \ldots$

Young's graph

Young's graph arises from containment of partitions:



Random partitions

Markov chains $X_0, X_1 \dots$ on this graph, such that

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Under this correspondence

$$p(\lambda) = \phi(s_\lambda)$$

and (Kerov-Vershik)

poles and zeroes $\{\alpha_i, \beta_i\} \leftrightarrow$ scaled lengths of i-th rows and columns

Two variations

Block-Toeplitz:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_0 & a_1 & a_2 & a_3 & \cdots \\ \cdots & 0 & b_0 & b_1 & b_2 & \cdots \\ \cdots & 0 & 0 & a_0 & a_1 & \cdots \\ \cdots & 0 & 0 & 0 & b_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}?$$

Finite:

$$\begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 1 & a_0 & a_1 & a_2 \\ 0 & 0 & 1 & a_0 & a_1 \\ 0 & 0 & 0 & 1 & a_0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}?$$

Consider the formal loop group $GL_n(\mathbb{R}((t)))$ consisting of (invertible) $n \times n$ matrices, whose entries are formal Laurent series. To each such matrix $X(t)$ we can associate an infinite periodic (block-Toeplitz) matrix $A(X)$:

$$\begin{pmatrix} 1+t^2 & 2+5t \\ -1-t & -4t^2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 5 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & -1 & 0 & 0 & -4 & 0 & 0 & \cdots \\ \cdots & 1 & 2 & 0 & 5 & 1 & 0 & \cdots \\ \cdots & -1 & 0 & -1 & 0 & 0 & -4 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Total positivity for loop groups (with Pavlo Pylyavskyy)

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A matrix $X(t) \in GL_n(\mathbb{R}((t)))$ is TNN if $A(X)$ is.
Note that $A(X(t)Y(t)) = A(X)A(Y)$.

Whirls and curls

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Which matrices play the role of poles and zeroes for $GL_n(\mathbb{R}((t)))$?

Whirls and curls

Whirls $M(\beta_i^{(1)}, \ldots, \beta_i^{(n)})$, and curls $N(\alpha_i^{(1)}, \ldots, \alpha_i^{(n)})$, depending on n (positive) parameters.
Let $n = 2$.

$$M(a,b) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & a & 0 & 0 & \cdots \\ \cdots & 0 & 1 & b & 0 & \cdots \\ \cdots & 0 & 0 & 1 & a & \cdots \\ \cdots & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$N(a,b) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & a & ab & a^2b & \cdots \\ \cdots & 0 & 1 & b & ab & \cdots \\ \cdots & 0 & 0 & 1 & a & \cdots \\ \cdots & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Unlike the factors $(1 + \alpha t)$ and $(1 + \beta t)$, two whirls $M(\alpha_1, \ldots, \alpha_n)$ and $M(\beta_1, \ldots, \beta_n)$ do not always commute, but satisfy a commutation relation involving a rational transformation of the parameters.

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Theorem (L.-Pylyavskyy)

The transformation $(\alpha, \beta) \mapsto (\alpha', \beta')$, where $M(\alpha)M(\beta) = M(\alpha')M(\beta')$ is the "birational R-matrix" for the symmetric power representation of $U_q(\hat{\mathfrak{sl}}_n)$.

The R-matrix $R : V \otimes W \cong W \otimes V$ interchanges factors in tensor products of representations of quantum groups.

Factorization theorem for loop groups

Theorem (L.-Pylyavskyy)

Every upper triangular $X \in GL_n(\mathbb{R}((t)))$ can be factorized as

$$\prod_{i=1}^{\infty} N(\alpha_i^{(1)}, \ldots, \alpha_i^{(n)}) \exp(Y(t)) \prod_{i=-\infty}^{-1} M(\beta_i^{(1)}, \ldots, \beta_i^{(n)})$$

for suitable positive parameters, where $Y(t)$ is entire.

Furthermore, the three factors are unique.

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Furthermore, the three factors are unique.

There is a ring LSym with a distinguished spanning set, called Loop symmetric functions, such that

$$\text{TNN points of loop group} \xleftrightarrow{1:1} \text{positive homomorphisms of LSym}$$

Infinite products of Chevalley generators

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This part of $GL_n(\mathbb{R}((t)))_{\geq 0}$ contains elements of the form

$$X = e_{i_1}(t_1)e_{i_2}(t_2)e_{i_3}(t_3)\cdots$$

where $\{e_i(t) \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ are the (affine) Chevalley generators.

$n = 3$

$$e_1(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad e_2(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \quad e_0(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ at & 0 & 1 \end{pmatrix}$$

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Very often these can also be written as

$$X = e_{j_1}(t'_1)e_{j_2}(t'_2)e_{j_3}(t'_3)\cdots$$

This leads to a notion of braid limits inside Coxeter groups.

Example of braid limit

Take $n = 3$, and the affine symmetric group with simple generators s_0, s_1, s_2 satisfying

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Theorem (L.-Pylyavskyy)

Can always end up at an infinite power of a Coxeter element. For $n = 3$: $(012)^\infty$, $(120)^\infty$, $(201)^\infty$, $(210)^\infty$, $(102)^\infty$, $(021)^\infty$

Finite Toeplitz matrices

What about finite totally nonnegative matrices of the form

$$M = \begin{pmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 1 & a_0 & a_1 & a_2 \\ 0 & 0 & 1 & a_0 & a_1 \\ 0 & 0 & 0 & 1 & a_0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ?$$

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This problem was studied by Rietsch.

Functions on finite Toeplitz matrices

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Theorem (Ginzburg, Peterson)

We have a canonical isomorphism

$$\mathcal{O}(X) \simeq H_*(\mathrm{Gr}_{SL_n}, \mathbb{C})$$

between the ring of functions $\mathcal{O}(X)$ and the homology of the affine Grassmannian $\mathrm{Gr}_{SL_n} = SL_n(\mathbb{C}((t)))/SL_n(\mathbb{C}[[t]])$.

Affine Grassmannian

1. The space Gr_{SL_n} is an ind-scheme, with distinguished subvarieties called Schubert varieties.

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giving $H_*(\mathrm{Gr}_{SL_n})$ a ring structure.
3. The ring $H_*(\mathrm{Gr}_{SL_n})$ contains a distinguished basis $\{\sigma_w\}$ called the Schubert basis:

$$H_*(\mathrm{Gr}_{SL_n}) = \oplus_w \mathbb{C} \cdot \sigma_w$$

Schubert positivity

Theorem (Rietsch, translated via the next theorem)

$M \in X(\mathbb{R})$ is "totally positive" $\Leftrightarrow \sigma_w(M) > 0$ for all w.

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Theorem (Peterson; L.-Shimozono)

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There is also a "parametrization" result which is of a flavor different to the Edrei-Thoma theorem.

k-Schur functions and cores

Theorem (L.)

The following sets are in bijection:

- *The totally nonnegative finite Toeplitz matrices.*

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- *Homomorphisms* $\phi : \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}] \to \mathbb{R}$ *such that* $\phi(s_\lambda^{(k)}) \geq 0$ *for each* k*-Schur function* $s_\lambda^{(k)}$*.*

$\mathbb{Z}[h_1, h_2, \ldots, h_{n-1}] \subset \mathrm{Sym}$ is generated by the first $n-1$ homogeneous symmetric functions.

$s_\lambda^{(k)}$ is the k-Schur function of Lapointe-Lascoux-Morse (with $k = n-1$, and $t = 1$) occurring in the study of Macdonald polynomials.

Theorem (L.)

There is an isomorphism $H_*(\mathrm{Gr}_{SL_n}) \cong \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}]$ *sending Schubert classes to* k*-Schur functions.*

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Theorem (L.)

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- *Extremal Markov chains on the graph of* n*-cores, such that the probability of a tableau only depends on its shape.*

The graph of n-cores

In Young's lattice, boxes are added one at a time. In the graph of n-cores, many boxes can be added at the same step.

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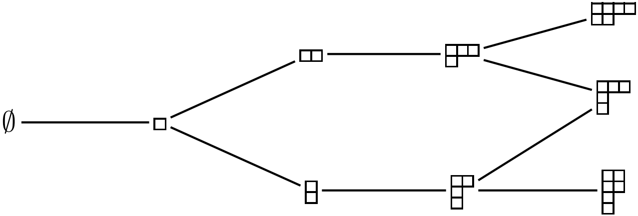
For $n = 3$



Schubert vs. total positivity via Geometric Satake

Why should Schubert positivity have anything to do with total positivity?

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$$\text{MV-cycles} \subset \mathrm{Gr}_G \leftrightarrow \text{weight vectors in irreps of } G^\vee$$

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This is consistent with Lusztig's definition of totally nonnegative elements as those that act positively on the canonical basis, and the general philosophy that MV-cycles are a geometric canonical basis.