Total positivity, crystals, and Whittaker functions

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Total positivity

A real matrix is totally nonnegative (TNN) if every minor is nonnegative.
A real matrix is totally positive (TP) if every minor is positive.

Example

$$M = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$$

is not TNN.
But

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 3 \end{pmatrix}$$

is TP.

Totally nonnegative part of $GL_n(\mathbb{R})$

$GL_n(\mathbb{R})_{\geq 0}$ = TNN part of GL_n

Theorem (Loewner-Whitney)

$GL_n(\mathbb{R})_{\geq 0}$ is the semigroup generated by $T_{>0}$ and positive Chevalley generators $x_i(a), y_i(a)$ with $a \geq 0$

$$\mathrm{diag}(t_1, t_2, t_3, t_4) = \begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{bmatrix} \qquad t_1, t_2, t_3, t_4 > 0$$

$$x_2(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad y_1(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus totally positive matrices can be row reduced via positive adjacent row operations.

Some notation

G = complex simple algebraic group (split over $\mathbb{R}$) $\qquad SL(n)$

B, B_- = opposite Borels $\qquad \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \quad \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$

U, U_- = unipotent subgroups $\qquad \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$

$T = B \cap B_-$ torus $\qquad \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$

I = vertex set of Dynkin diagram $\qquad \{1, 2, \ldots, n-1\}$

$x_i : \mathbb{C} \to U, i \in I$ one parameter subgp

$y_i : \mathbb{C} \to U_-, i \in I$ one parameter subgp

Lusztig's generalization

Definition (Totally nonnegative part of G)

$G_{\geq 0}$ is the semigroup generated by $T_{>0}$ and $x_i(a), y_i(a)$ for $a > 0$.

$$T_{>0} = \langle \nu(a) \mid a \in \mathbb{R}_{>0} \rangle \subset T$$

as ν ranges over cocharacters $\nu : \mathbb{C}^* \to T$

Relations for semigroup generators

$$x_1(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad x_2(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1(a)x_2(b)x_1(c) = x_2(bc/(a+c))x_1(a+c)x_2(ab/(a+c))$$

gives a subtraction-free birational transformation of tori $(\mathbb{C}^*)^3 \to (\mathbb{C}^*)^3$

$$(a, b, c) \longmapsto (bc/(a+c), a+c, ab/(a+c))$$

Similar positive birational transformations play an important role for example in

- Fomin and Zelevinsky's cluster algebras
- Fock and Goncharov's coordinates for Teichmuller spaces
- Berenstein and Kazhdan's geometric crystals

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Subtraction-free rational formulae can be tropicalized by

$$+ \longmapsto \min, \quad \times \longmapsto +, \quad \div \longmapsto -$$

giving a map of lattices $\mathbb{Z}^3 \to \mathbb{Z}^3$

$$(A, B, C) \longmapsto (B + C - \min(A, C), \min(A, C), A + B - \min(A, C))$$

Occurs in parametrizations of canonical bases. Industry of lifting piecewise-linear formulae to subtraction-free rational formulae.

Kashiwara's crystal graphs

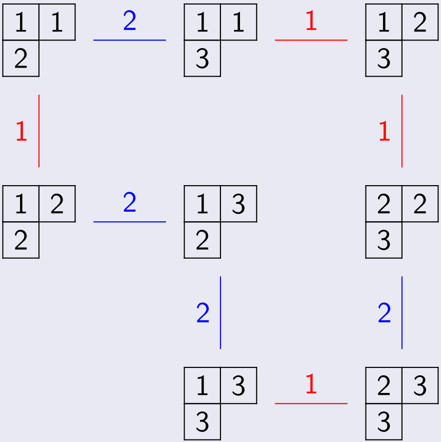
Kashiwara's crystal graphs are combinatorial models for highest weight representations of a semisimple complex Lie algebra $\mathfrak{g}$.

$$\begin{aligned} \mathfrak{g} &= \mathrm{Lie}\ G \\ V(\lambda) &= \text{ finite dimensional irrep with highest weight } \lambda \\ B(\lambda) &= \text{ Kashiwara's crystal graph} \end{aligned}$$

Kashiwara's crystal graphs

weight function $\mathrm{wt} : B(\lambda) \to \mathbb{Z}^r$

Example: $B(2,1)$ for 8-dim adjoint representation V of $\mathfrak{sl}_3$



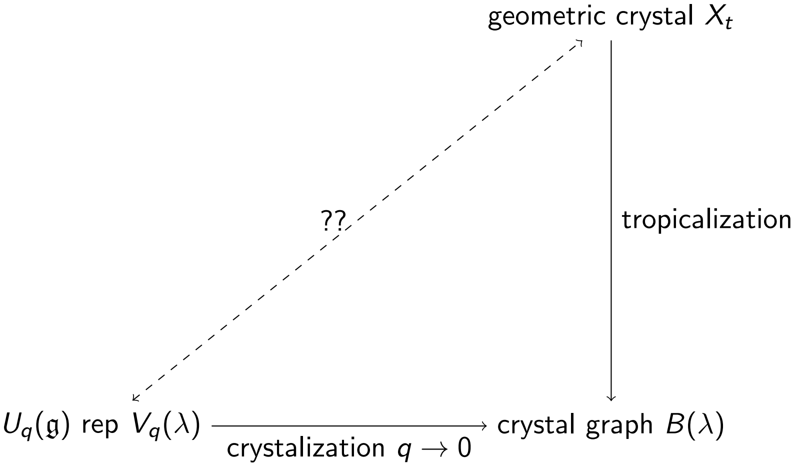
character $\chi_V(x_1, x_2, x_3)$

$$=\sum_{b\in B(\lambda)} x^{\mathrm{wt}(b)}$$

$$=x_1^2x_2 + x_1^2x_3 + 2x_1x_2x_3 +$$

$$x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Crystalization vs. tropicalization



Geometric crystals

Berenstein and Kazhdan defined geometric crystals, extending the work of Berenstein and Kirillov, and Berenstein and Zelevinsky.

Definition

A geometric crystal is a tuple $(X, \gamma, \mathcal{F}, e_i, \varepsilon_i, \varphi_i)$, where X is an irreducible variety over $\mathbb{C}$, equipped with

- a rational map $\gamma : X \to T$, called the weight;
- a rational function $\mathcal{F} : X \to \mathbb{C}$, called the decoration;
- rational actions $e_i : \mathbb{C}^* \times X \to X$ for $i \in I$ that we won't discuss;
- rational functions ε_i and φ_i for $i \in I$ that we won't discuss.

satisfying a list of axioms.

The geometric crystal with highest weight t

Fix

$w_0 \in G$ rep. of longest element in Weyl group $\quad w_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}$

$\chi : U \to \mathbb{C}$ nondegenerate character $\quad \chi \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = a + c$

Definition

For $t \in T$, the geometric crystal with highest weight t is

$$X_t := B_- \cap U\, t\, w_0\, U$$

where

- $\gamma(x) = x \mod U_- \in B_-/U_- \simeq T$
- $\mathcal{F}(u_1 t\, w_0 u_2) = \chi(u_1) + \chi(u_2)$

Example for $G = GL_3$

$U \subset GL_3$ is parametrized birationally by the map

$$(a,b,c) \in (\mathbb{C}^\times)^3 \longmapsto \begin{bmatrix} 1 & b & bc \\ & 1 & a+c \\ & & 1 \end{bmatrix} = x_2(a)x_1(b)x_2(c) =: u_1.$$

Let

$$t = \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix} \qquad w_0 = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix}$$

Then X_t is parametrized by $(a,b,c) \longmapsto$

$$x = \begin{bmatrix} bct_3 & & \\ (a+c)t_3 & at_2/c & \\ t_3 & t_2/c & t_1/ab \end{bmatrix} = u_1 t w_0 \begin{bmatrix} 1 & t_2/ct_3 & t_1/abt_3 \\ & 1 & (a+c)t_1/abt_2 \\ & & 1 \end{bmatrix}$$

γ and $\mathcal{F}$

We read off that

$$\gamma(x) = \begin{bmatrix} bct_3 & 0 & 0 \\ 0 & \frac{at_2}{c} & 0 \\ 0 & 0 & \frac{t_1}{ab} \end{bmatrix}$$

and

$$\begin{aligned} \mathcal{F}(x) &= (a+c) + b + \frac{t_2}{ct_3} + \frac{(a+c)t_1}{abt_2} \\ &= a + b + c + \frac{t_2}{ct_3} + \frac{t_1}{bt_2} + \frac{ct_1}{abt_2} \end{aligned}$$

using the $\chi : U \to \mathbb{C}$ from before.

Berenstein and Kazhdan's theorem

$$\mathbb{C}^* \longmapsto \mathbb{Z}$$

$$T \longmapsto \text{lattice of characters}$$

$$+ \longmapsto \min, \quad \times \longmapsto +, \quad \div \longmapsto -$$

$$(a, b, c) \longmapsto (A, B, C)$$

$$(t_1, t_2, t_3) \longmapsto (\lambda_1, \lambda_2, \lambda_3)$$

Theorem (Berenstein-Kazhdan)

$$X_t \textit{ tropicalizes to } B(\lambda).$$

In particular, there is a bijection

$$\{(A, B, C) \in \mathbb{Z}^3 \mid \mathrm{trop}\mathcal{F} \geq 0\} \longrightarrow B(\lambda)$$

$$\mathrm{trop}\gamma \longmapsto \mathrm{wt}$$

Gelfand-Tsetlin patterns

Tropicalizing, the condition $\mathrm{trop}(\mathcal{F}) \geq 0$ becomes

$$\min(A, B, C, \lambda_2 - C - \lambda_3, \lambda_1 - B - \lambda_2, C + \lambda_1 - A - B - \lambda_2) \geq 0$$

which can be arranged into a Gelfand-Tsetlin pattern

$$\begin{array}{ccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 \\
 & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & \\
 & & \lambda_1 - B & & & & \lambda_2 - C & & \\
 & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & \\
 & & & & \lambda_1 - A - B & & & &
\end{array}$$

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 & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & \\
 & & & & \lambda_1 - A - B & & & &
\end{array}$$

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2 & 3 \\
\hline
2 & 2 & 3 & 3 & \multicolumn{1}{c}{} \\
\cline{1-4}
3 & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}
\longmapsto
\begin{array}{ccc}
5 & 4 & 1
\end{array}$$

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& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & & \lambda_1 - A - B & & & &
\end{array}$$

$$\begin{array}{|c|c|c|c|l}
\hline
1 & 1 & 1 & 2 & 3 \\
\hline
2 & 2 & \multicolumn{1}{|c}{3} & 3 & \\
\cline{1-2}
\multicolumn{1}{c}{3} & & & &
\end{array}
\quad \longmapsto \quad
\begin{array}{ccccc}
5 & & 4 & & 1 \\
& 4 & & 2 &
\end{array}$$

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& & \lambda_1 - B & & & & \lambda_2 - C & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & & \lambda_1 - A - B & & & &
\end{array}$$

$$\begin{array}{ccccc}
\boxed{1} & \boxed{1} & \boxed{1} & 2 & 3 \\
2 & 2 & 3 & 3 & \\
3 & & & &
\end{array}
\quad \longmapsto \quad
\begin{array}{ccccc}
5 & & 4 & & 1 \\
& 4 & & 2 & \\
& & 3 & &
\end{array}$$

What is the geometric character?

Also it is easy to check that

$$\mathrm{trop}\gamma_3(A,B,C) = \#ones$$
$$\mathrm{trop}\gamma_2(A,B,C) = \#twos$$
$$\mathrm{trop}\gamma_1(A,B,C) = \#threes$$

The character χ_V of V is the generating function of all elements in the crystal graph of V. (For $G = GL_n$, this is a Schur polynomial, e.g. $x_1^2x_2 + x_1^2x_3 + 2x_1x_2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$.)

$$\chi_{V(\lambda)} = \sum_{b \in B(\lambda)} x^{\mathrm{wt}(b)}$$

Question

What is the character of a geometric crystal?

Jacquet's Whittaker functions

$G(\mathbb{R})$ split real group
$K \subset G(\mathbb{R})$ maximal compact subgroup
$U(\mathbb{R})$ real points of unipotent

Jacquet's Whittaker functions for real groups

A Whittaker function is a smooth function $\psi \in C^\infty(G(\mathbb{R})/K)$ which

1. transforms as $\psi(uz) = \chi(u)\psi(z)$ for $u \in U(\mathbb{R})$;
2. is an eigenfunction for the left-invariant differential operators on $G(\mathbb{R})/K$.

Kostant's Whittaker functions

The quantum Toda lattice is a quantum integrable system, that is, a system of commuting differential operators, on T with quantum Hamiltonian

$$H = \frac{1}{2}\Delta - \chi_i \sum_{i \in I} \alpha_i(t)$$

where $\alpha_i : T \to \mathbb{C}^*$ are the simple roots of G, and $\chi_i \in \mathbb{C}$ depends on $\chi : U \to \mathbb{C}$. (Can take $\chi_i = 1$.)

Kostant's Whittaker functions

A Whittaker function is a smooth function $\psi(t) \in C^\infty(T)$ or $\psi(t) \in C^\infty(T(\mathbb{R}))$ that is an eigenfunction of the quantum Toda lattice.

Same as Jacquet's version, via quantum Hamiltonian reduction.

Integral formula for Whittaker functions

Let

$$\omega_t = \frac{da_1}{a_1} \wedge \frac{da_2}{a_2} \wedge \cdots \wedge \frac{da_r}{a_r}$$

be the canonical form on X_t, where $(a_1, \ldots, a_r) \in (\mathbb{C}^*)^r \to X_t$ is a positive parametrization.

Theorem (L., similar statement by Chhaibi)

Let $\lambda : T \to \mathbb{C}^*$ be a character of T. The integral function

$$\psi_\lambda(t) = \int_{(X_t)_{\geq 0} \subset X_t} \lambda(\gamma(x)) e^{-\mathcal{F}(x)} \omega_t.$$

is a Whittaker function on $T_{>0}$, with infinitesimal character $\lambda - \rho$.

$$\int_{(X_t)_{\geq 0}} = \int_{\mathbb{R}^r_{>0}}$$

Whittaker function as integral over geometric crystal

- Givental [1997]: gave this integral formula for $\lambda = 1$ and $G = GL_n$
- Joe and Kim [2003]: generalized Givental's work to arbitrary λ
- Gerasimov, Kharchev, Lebedev, Oblezin [2006] and Gerasimov, Lebedev, Oblezin [2012]: studied these integral formulae, and proved them for classical groups
- Rietsch [2008]: Conjectured the formula for arbitrary G (without geometric crystals).
- Rietsch [2012]: Proved her conjecture for $\lambda = 1$.
- Chhaibi: independently found the same formula and proved that it is an eigenfunction of the quantum Toda Hamiltonian. Probabilistic interpretation following O'Connell's work.

Whittaker functions as geometric characters

Schur polynomials $s_\mu(x_1, x_2, \ldots, x_n) = \sum_T x^{\mathrm{content}(T)}$

are generating functions over Young tableaux with a fixed shape μ.

Irreducible characters $\chi_\mu = \sum_{b \in B(\mu)} e^{\mathrm{weight}(b)}$

are weight-generating functions over crystals.

Whittaker functions $\psi_\lambda(t) = \int_{(X_t)_{\geq 0} \subset X_t} \lambda(\gamma(x)) e^{-\mathcal{F}(x)} \omega_t$.

are integrals over the totally positive part of a geometric crystal. The function $\mathcal{F}(x)$ takes the place of the "semistandard" condition on tableaux.

Birational combinatorics (much inspired by total positivity)

Piecewise - linear functions $\longrightarrow$ Subtraction-free formulae

- Formulae for Littlewood-Richardson coefficients (Berenstein-Zelevinsky)
- Lascoux-Schützenberger charge, or energy functions (L.-Pylyavskyy)

Combinatorial algorithms $\longrightarrow$ Subtraction-free birational maps

- Cluster algebra dynamical systems (many authors)
- Geometric RSK (Kirillov, Noumi-Yamada)
- Birational promotion, rowmotion, ... (Einstein-Propp, Grinberg-Roby)

Formal power series $\longrightarrow$ Formal integrals??

Replace summation over $(\mathbb{Z}_{\geq 0})^{\ell}$ by formal integral over $\mathbb{R}^{\ell}_{>0}$.

Three other directions

Non-archimedean case = q-analogue

Replacing $G(\mathbb{R})$ by $G(\mathbb{Q}_p)$, we obtain the Casselman-Shalika formula, which relates the non-Archimedean Whittaker function with the Weyl character formula. Many recent generalizations e.g. to metaplectic groups.

Mirror symmetry for flag varieties

In the setting of Givental and Rietsch's mirror conjecture, our theorem says that $X_t =$ mirror family to $G^\vee/B^\vee$ via a theorem of B. Kim.

Directed polymers

O'Connell showed that the law of the partition function of a Brownian directed polymer model could be described in terms of Whittaker functions for GL_n.
The Whittaker measure is an analogue of the Schur measure on partitions (Corwin, O'Connell, Seppäläinen, Zygouras).