Totally nonnegative Grassmannian and the amplituhedron

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November 2014

TNN Grassmannian

Take integers $1 \leq k \leq n$. The Grassmannian $\mathrm{Gr}(k,n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$.

$$X = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$$

Definition (Totally nonnegative Grassmannian (Postnikov))

The totally nonnegative Grassmannian $\mathrm{Gr}(k,n)_{\geq 0}$ is the locus in the real Grassmannian representable by X such that all $k \times k$ minors (Plücker coordinates) are nonnegative.

Lusztig defined the TNN parts of arbitrary G/P.

$$\text{Lusztig's } \mathrm{Gr}(k,n)_{\geq 0} = \overline{GL(n)_{>0} \cdot b}$$

where $b = \mathrm{span}(e_1, e_2, \ldots, e_k)$ is a particular basepoint. The two definitions are (not obviously) equivalent.

Amplituhedron

Definition (Arkani-Hamed and Trnka's amplituhedron)

The amplituhedron $A(k,n,d)$ in $\mathrm{Gr}(k,d)$ is the image of $\mathrm{Gr}(k,n)_{\geq 0}$ under a (positive) linear map $Z:\mathbb{R}^n \to \mathbb{R}^d$ inducing $Z_{\mathrm{Gr}}:\mathrm{Gr}(k,n) \to \mathrm{Gr}(k,d)$.

- Z_{Gr} is not defined everywhere in $\mathrm{Gr}(k,n)$. If Z is positive, then it is well-defined on $\mathrm{Gr}(k,n)_{\geq 0}$.
- This is only the "tree" amplituhedron. There is also a "loop" amplituhedron.

Polytopes to amplituhedra

$Gr(1, n)_{\geq 0}$ is a simplex

$\mathrm{Gr}(1, n)_{\geq 0} = \{(x_1, x_2, \ldots, x_n) \neq \mathbf{0} \mid x_i \in \mathbb{R}_{\geq 0}\}$ modulo scaling

which can be identified with the simplex

$\Delta_{n-1} := \{(a_1, a_2, \ldots, a_n) \mid a_i \in [0, 1]$ and $a_1 + a_2 + \cdots + a_n = 1\}$.

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Polytopes are images of simplices

A convex polytope in $\mathbb{R}^d$ with vertices $v_1, v_2, \ldots, v_n$ is the image of

$$\Delta_{n-1} \subset \mathbb{R}^n$$

under a linear map $Z : \mathbb{R}^n \to \mathbb{R}^d$ where

$$Z(e_i) = v_i.$$

Amplituhedron form

Amplituhedron form

The amplituhedron form ω is a meromorphic top form on $\mathrm{Gr}(k, d)$ with logarithmic singularities on the codimension one faces of the amplituhedron. (The Zariski-closures of such faces are complex divisors in $\mathrm{Gr}(k, d)$.)

There are many meromorphic forms satisfying this condition.

Problem

Give a good "formula" for ω.

But first, why do the physicists care?

$\mathcal{N} = 4$ $D = 4$ super Yang-Mills

What is this quantum field theory?

$\mathcal{N} = 4$ $D = 4$ Yang-Mills for us means:

- spacetime is $M = \mathbb{R}^4$ with the usual Lorentz metric
- consider a pure gauge theory with gauge group $SU(m)$ (classical fields in this theory are connections on a $SU(m)$-bundle over M)
- add 4 super symmetries (the maximal amount) to this quantum field theory. The resulting theory has 16 types of particles.

Amplitudes

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$$A_n = A_n^{\text{tree}} + A_n^{\text{1-loop}} + \cdots$$

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where A_n^{tree} only considers Feynman diagrams without loops. A trick called color-ordering allows one to write

$$A_n^{\text{tree}} = (\text{group theory factor})\mathcal{A}_n^{\text{tree}},$$

so that $\mathcal{A}_n^{\text{tree}}$ has no dependence on the gauge group.

Tree amplitude as a rational form

- In the right coordinates, $\mathcal{A}_n^{\mathrm{tree}}$ is much simpler than expected (often just a simple rational function)!
- It exhibits surprising symmetries: besides the usual superconformal symmetry, there is a dual superconformal symmetry which glue to give a Yangian $Y(\mathfrak{psu}(2,2|4))$ symmetry.

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In the amplituhedron setting, this data is stored in:

- the $(k+4) \times n$ matrix Z, together with
- a point $Y \in \mathrm{Gr}(k, k+4)$, which roughly speaking, says which of the 4 rows give the momenta, and the other k deal with supersymmetry.

$$\mathcal{A}_n^{\text{tree}} = \mathcal{A}_n^{\text{tree}}(Y,Z) = \text{"amplituhedron form"} = \omega_{SYM}(Y,Z)$$

Amplituhedron form in the polytope case

We have

$$Z_{\mathrm{Gr}} : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, d)$$

where $d = k + 4$ is the physical case. We want to describe a meromorphic top form ω on $\mathrm{Gr}(k, d)$, which depends on Z.

Polytope case

For $k = 1$, and $Z : \mathbb{P}^{n-1} \to \mathbb{P}^{d-1}$, we have

$$\omega = \mathcal{L}(\chi_{P^\vee})$$

where

- $P^\vee$ is the dual polytope to $P = Z(\Delta_{n-1})$,
- $\chi_{P^\vee}$ is its characteristic function, and
- $\mathcal{L}$ is the Laplace transform, sending piecewise-linear functions in $\mathbb{P}^{d-1}$ to rational functions in $\mathbb{P}^{d-1}$.

No such simple formula is known for Grassmannians.

Positroid stratification

$$X = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

Extend periodically via $v_{i+n} = v_i$, to get

$$\ldots, v_{-1}, v_0, v_1, v_2, v_3 \ldots, \in \mathbb{C}^k.$$

Define $f_X : \mathbb{Z} \to \mathbb{Z}$ by

$$f_X(i) = \min\{j \geq i \mid v_i \in \mathrm{span}(v_{i+1}, v_{i+2}, \ldots, v_j)\}.$$

Bounded affine permutations

Theorem (Postnikov)

f_X is a (k, n) bounded affine permutation:

- $f_X : \mathbb{Z} \to \mathbb{Z}$ *is a bijection,*
- $f_X(i + n) = f_X(i) + n$,
- $i \leq f_X(i) \leq i + n$,
- $\sum_{i=1}^{n}(f(i) - i) = kn$,

and these are exactly the functions that occur.

For $f \in \mathrm{Bound}(k, n)$, set

$$\mathring{\Pi}_f := \{X \in \mathrm{Gr}(k, n) \mid f_X = f\}$$

and define the positroid variety

$$\Pi_f = \overline{\mathring{\Pi}_f}.$$

These subvarieties form a stratification of $\mathrm{Gr}(k, n)$.

Some facts

- $\mathring{\Pi}_f \cap \mathrm{Gr}(k,n)_{\geq 0} \simeq \mathbb{R}^r_{>0}$, where $r = \dim_{\mathbb{C}} \Pi_f$ (Postnikov, Lusztig, Rietsch)
- Π_f is irreducible, normal, Cohen-Macaulay, and has rational singularities (Knutson.-L.-Speyer)

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Other characterizations:

- Π_f can be defined (scheme-theoretically) as the intersection of cyclically rotated Schubert varieties (Postnikov, Knutson.-L.-Speyer)
- Π_f are exactly the torus-orbits of symplectic leaves for the natural Poisson structure on $\mathrm{Gr}(k, n)$ (Goodearl-Yakimov)
- Π_f are exactly the compatibly Frobenius split subvarieties of the Grassmannian with respect to the natural Frobenius splitting (Knutson.-L.-Speyer)

Also relations to: cluster algebras, mirror symmetry, KP solitons, symmetric functions, affine and quantum Schubert calculus,...

Canonical top form of a positroid variety

Define the boundary

$$\partial \Pi_f = \bigcup_{g \neq f \mid \Pi_g \subset \Pi_f} \Pi_g$$

and the canonical form

$$\omega_{\Pi_f} := \text{unique nonzero top form with simple poles along } \partial \Pi_f$$

on Π_f.

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Can also be defined with a birational parametrization similar to

$$(a, b, c) \mapsto x_1(a)x_2(b)x_1(c) = \begin{pmatrix} 1 & a+c & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \qquad \omega = \frac{da}{a}\frac{db}{b}\frac{dc}{c}$$

from the previous lecture. (Yet another way: use cluster structure.)

Canonical forms of amplituhedron varieties

Define the amplituhedron variety

$$Y_f := \overline{Z_{\mathrm{Gr}}(\Pi_f)}$$

and its canonical form as a pushforward

$$\omega_{Y_f} := \mathrm{trace}_{Z_{\mathrm{Gr}}}(\omega_{\Pi_f}).$$

- If $\dim(Y_f) < \dim(\Pi_f)$, then $\omega_{Y_f} = 0$.
- If $\dim(Y_f) = \dim(\Pi_f)$, then ω_{Y_f} is obtained by summing over finitely many preimages (counting these is a Schubert calculus problem).

Physicists' idea

ω_{SYM} is the sum of ω_{Y_f} over a triangulation of the amplituhedron.

Expressing the amplitude as a sum over positroid varieties was also studied earlier by Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka.

Grassmannian polytopes

Question

How do we describe Y_f and ω_{Y_f} directly?

The faces of polytopes, and of their triangulations, are all cut out by linear functions.

This is not true for the amplituhedron varieties; some are cut out by higher degree polynomials in the Plücker coordinates (the $k \times k$ minors of $\mathrm{Gr}(k, n)$).

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What class of functions on the Grassmannian should we replace linear functions with, to get a good theory of Grassmannian polytopes?

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Possible Idea

Use Lusztig's canonical bases, or Kashiwara's global bases.

Amplituhedron varieties and canonical bases

(Dual) canonical bases elements are distinguished functions (sections of some line bundle) on the Grassmannian, which are higher degree, but possess positivity properties generalizing those of Plücker coordinates. For example, these bases elements are positive on $\mathrm{Gr}(k,n)_{\geq 0}$.

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Y_f depends on Z, so to make sense of this, one actually works with a "universal amplituhedron variety" $\mathcal{Y}_f \subset \mathrm{Gr}(k,d) \times \mathrm{Mat}(n,d)$.

For the rest of the talk, I will focus on explaining a simpler relation between Π_f and the dual canonical basis.

Ideal of positroid variety

$$R(k,n) = \bigoplus_m R(k,n)_m = \text{homogeneous coordinate ring of } \mathrm{Gr}(k,n)$$

Example

R(2,4) $=$

$$\mathbb{C}[\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}]/(\Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23} - \Delta_{12}\Delta_{34}).$$

$R(2,4)_1$ is 6-dimensional spanned by $\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}$.
$R(2,4)_2$ is $\binom{6}{2} + 6 - 1 = 20$-dimensional.

Ideal of positroid variety

$$R(k,n) = \bigoplus_m R(k,n)_m = \text{homogeneous coordinate ring of } \mathrm{Gr}(k,n)$$

As a $GL(n)$-representation,

$$R(k,n)_m = V(m\omega_k)^*$$

where $V(m\omega_k)$ is the finite-dimensional irreducible representation, and ω_k is the k-th fundamental weight. Thus $V(\omega_k) \simeq \Lambda^k \mathbb{C}^n$. In combinatorial language, $m\omega_k$ is a $k \times m$ rectangle.
The $GL(n)$-module $V(m\omega_k)$ has a distinguished basis

$$\{G(T) \mid T \in B(m\omega_k)\}$$

of canonical basis elements (evaluated at $q = 1$), constructed by Lusztig (another construction by Kashiwara).

The homogeneous ideal of a positroid variety

$$\mathcal{I}(\Pi_f) = \text{homogeneous ideal of } \Pi_f$$
$$\mathbb{C}[\Pi_f] = \text{homogeneous coordinate ring of } \Pi_f$$

Theorem (L.)

Both $\mathcal{I}(\Pi_f)$ and $\mathbb{C}[\Pi_f]$ are spanned by dual canonical basis elements.

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Theorem (L.)

Both $\mathcal{I}(\Pi_f)$ and $\mathbb{C}[\Pi_f]$ are spanned by dual canonical basis elements. More precisely,

- $\mathcal{I}(\Pi_f)_m \subset R(k,n)_m = V(m\omega_k)^*$ *is isomorphic to* $V_f(m\omega_k)^\perp$ *and has a basis given by* $\{G(T)^* \mid T \notin B_f(m\omega_k)\}$.
- $\mathbb{C}[\Pi_f]_m$ *is spanned by (images of)* $\{G(T)^* \mid T \in B_f(m\omega_k)\}$.

$$B_f(m\omega_k) = \text{cyclic Demazure crystal } \subset B(m\omega_k)$$

which indexes a basis of

$$V_f(m\omega_k) = \text{cyclic Demazure module } \subset V(m\omega_k),$$

defined to be the intersection of n cyclically rotated Demazure submodules of $V(m\omega_k)$.

These constructions seem to be of independent interest.

The crystal $B(m\omega_k)$

$B(m\omega_k)$ is the set of semistandard tableaux of shape $k \times m$, filled with the numbers $1, 2, \ldots, n$.

1	1	3	4	4
2	3	4	5	5
4	4	6	6	6

The set $B(m\omega_k)$ has a remarkable combinatorial operation called promotion, corresponding to the cyclic symmetry of the Grassmannian.

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The set $B(m\omega_k)$ has a remarkable combinatorial operation called promotion, corresponding to the cyclic symmetry of the Grassmannian.

$$\chi\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 3 & 4 & 4 \\ \hline 2 & 3 & 4 & 5 & 5 \\ \hline 4 & 4 & 6 & 6 & 6 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 4 & 4 & 5 & 5 \\ \hline 5 & 5 & 5 & 6 & 6 \\ \hline \end{array}$$

In fact, $B(m\omega_k)$ is a Kirillov-Reshetikhin crystal. They correspond to very special finite-dimensional representations of the Yangian.

Demazure submodules

$V(m\omega_k)$ has a highest weight vector v_+. We have

$$v_+ = G\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 & 3 \\ \hline \end{array}\right)$$

Definition

An extremal weight vector of $V(m\omega_k)$ is a vector v_I in the orbit $S_n \cdot v_+$, where $I \subset \{1, 2, \ldots, n\}$ is a k-element subset.

Let $\mathfrak{b}$ be the Lie algebra of the upper triangular matrices $B \subset GL(n)$, and $U(\mathfrak{b})$ be its universal enveloping algebra.

Definition

A Demazure submodule of $V(m\omega_k)$ is $V_I(m\omega_k) := U(\mathfrak{b}) \cdot v_I$.

The Demazure submodule $V_I(m\omega_k)$ has its own Demazure crystal $B_I(m\omega_k) \subset B(m\omega_k)$, and

$$V_I(m\omega_k) = \mathrm{span}(G(T) \mid T \in B_I(m\omega_k)) \subset V(m\omega_k).$$

Cyclic Demazure modules and crystals

Definition

For $k \in \mathrm{Bound}(k, n)$, the cyclic Demazure module is the intersection

$$V_f(m\omega_k) := V_{I_1}(m\omega_k) \cap \chi(V_{I_2}(m\omega_k)) \cap \cdots \cap \chi^{n-1}(V_{I_n}(m\omega_k))$$

where $(I_1, I_2, \ldots, I_n)$ is the Grassmann-necklace (appropriately shifted) of $f \in \mathrm{Bound}(k, n)$. The cyclic Demazure crystal is the intersection

$$B_f(m\omega_k) := B_{I_1}(m\omega_k) \cap \chi(B_{I_2}(m\omega_k)) \cap \cdots \cap \chi^{n-1}(B_{I_n}(m\omega_k)).$$

Theorem (L.)

$V_f(m\omega_k)$ has a basis given by $\{G(T) \mid T \in B_f(m\omega_k)\}$.

Geometry vs. representation theory vs. combinatorics

Geometry	Representation theory	Combinatorics
Grassmannian	Highest weight irreducible $V(m\omega_k)$	$B(m\omega_k)$
Schubert variety	Demazure module $V_I(m\omega_k)$	$B_I(m\omega_k)$
Positroid variety	cyclic Demazure module $V_f(m\omega_k)$	$B_f(m\omega_k)$

The cyclic symmetry is key for the last line, and isn't present for other G/P-s, except (co)minuscule ones where there is an analogue of promotion.

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Higher degree matroids

For $m = 1$, these crystals are matroids. Namely,

$$B(\omega_k) \simeq \text{uniform matroid}$$
$$B_I(\omega_k) \simeq \text{Schubert matroid}$$
$$B_f(\omega_k) \simeq \text{positroid}$$