Laurent phenomenon algebras

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August 3, 2012

This talk is based on joint work with P. Pylyavskyy.

What is a cluster algebra/Laurent phenomenon algebra?

A Laurent phenomenon algebra is a commutative algebra $\mathcal{A}$ with a distinguished set of generators, called cluster variables.

The cluster variables are arranged into collections $\{x_1, x_2, \ldots, x_n\}$ called clusters. Each cluster forms a transcendence basis for $\mathrm{Frac}(\mathcal{A})$ over the coefficient ring R.

The clusters are connected by mutation: for each cluster $\{x_1, x_2, \ldots, x_n\}$ and each $i \in \{1, 2, \ldots, n\}$, there is an adjacent cluster $\{x_1, x_2, \ldots, x_i', \ldots, x_n\}$ where the new cluster variable x_i' and the old one are related by

$x_i x_i' =$ exchange binomial for a cluster algebra

$x_i x_i' =$ exchange Laurent polynomial for a Laurent phenomenon algebra

where the RHS is a Laurent polynomial in $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ with coefficients in R.

Seeds

A Laurent phenomenon algebra is (essentially) completely determined by any one cluster and its exchange polynomials, called a seed:

$$(\mathbf{x}, \mathbf{F}) = (\{x_1, x_2, \ldots, x_n\}, \{F_1, F_2, \ldots, F_n\})$$

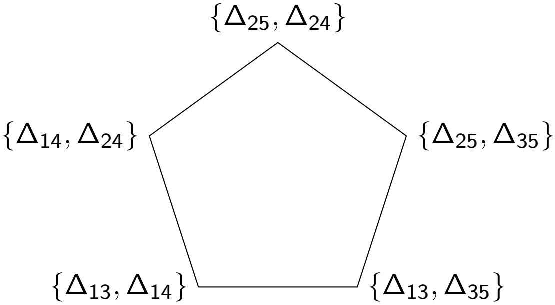
Mutation at i takes a seed $(\mathbf{x}, \mathbf{F})$ to a seed $(\mathbf{x}', \mathbf{F}')$:

1. We have $x_j' = x_j$ for all $j \neq i$.
2. But potentially F_j' differs from F_j for all $j \neq i$.

There is a (nearly) deterministic algorithm for producing $(\mathbf{x}', \mathbf{F}')$ from $(\mathbf{x}, \mathbf{F})$, and all seeds are assumed to be connected by mutation.

The Grassmannian $Gr(2, 5)$

Consider the homogeneous coordinate ring of the Grassmannian $Gr(2, 5)$, which is a six-dimensional projective variety. It has $\binom{5}{2} = 10$ Plucker coordinates $\Delta_{i,j}$ satisfying the Plucker relations. It can be arranged into a cluster algebra of rank 2 with 5 clusters:



where the coefficient ring is $R = \mathbb{C}[\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{45}, \Delta_{15}]$ and the exchange relations all look like

$$\Delta_{14}\Delta_{35} = \Delta_{15}\Delta_{34} + \Delta_{13}\Delta_{45}$$

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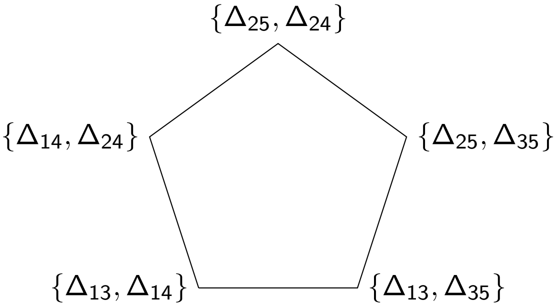

Note the combinatorics of a pentagon (dimension two associahedron) appearing out of a purely algebraic construction.

Combinatorics and cluster algebras

Cluster algebras have applications in

- total positivity
- coordinate rings of flag varieties and other Lie-theoretic varieties
- representation theory of quivers
- Poisson geometry
- Teichmüller theory
- integrable systems
- Donaldson-Thomas invariants
- ...

Combinatorics and cluster algebras

For a combinatorialist, we may think of cluster algebras as a machine which generates, and can be used to study:

- combinatorial recurrences: octahedron recurrence, Y-systems
- certain instances of the (positive) Laurent phenomenon
- polytopes known as generalized associahedra
- Catalan-style combinatorics
- combinatorics associated to planar networks and total positivity

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All of these features extend to our new Laurent phenomenon algebras. In some cases the combinatorial phenomenon had already been studied but the algebraic framework was unknown; in other cases we obtain new combinatorial phenomenon.

More careful definition

Let $(\{x_1, x_2, \ldots, x_n\}, \{F_1, \ldots, F_n\})$ be a seed. Then the mutated seed $(\mathbf{x}', \mathbf{F}') = \mu_i(\mathbf{x}, \mathbf{F})$ has exchange polynomials F_j' roughly given by the following procedure.

1. Substitution step: Let
$$G_j = F_j\big|_{x_i \leftarrow \frac{\hat{F}_i|_{x_j \leftarrow 0}}{x_i'}}$$
2. Cancellation step: Next remove all common factors between G_j and $\hat{F}_i|_{x_j \leftarrow 0}$ from G_j, to obtain H_j.
3. Normalization step: Finally, normalize H_j using a Laurent monomial to get an irreducible polynomial F_j'.

The tricky step is the cancellation step. For a cluster algebra this cancellation is always a monomial, and so can be absorbed into the last step.

Laurent phenomenon

Theorem (L.-Pylyavskyy)

Let $(\mathbf{x}, \mathbf{F})$ and $(\mathbf{y}, \mathbf{G})$ be two seeds in a Laurent phenomenon algebra. Then each y_i is a Laurent polynomial in the x_i.

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From the beginning of the theory, Fomin and Zelevinsky were aware that this Laurent phenomenon held beyond the cluster setting, including for recurrences such as the Gale-Robinson and Somos sequences, and the cube recurrence.

Gale-Robinson LP algebra

One case of the Gale-Robinson recurrence:

$$y_i y_{i+6} = y_{i+3}^2 + y_{i+2} y_{i+4} + y_{i+1} y_{i+5}.$$

Laurent phenomenon for Gale-Robinson sequence:

Theorem (Fomin-Zelevinsky)

$y_7, y_8, \ldots$ are Laurent polynomials in $\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}, y_4^{\pm 1}, y_5^{\pm 1}, y_6^{\pm 1}]$

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$$y_7 = \frac{y_2 y_6 + y_3 y_5 + y_4^2}{y_1}$$

$$y_8 = \frac{y_3 y_7 + y_4 y_6 + y_5^2}{y_2} = \cdots$$

$$y_{13} = \frac{y_8 y_{12} + y_9 y_{10} + y_{11}^2}{y_7} = ???$$

Gale-Robinson LP algebra

Start with the seed

$$\{(y_1, y_4^2 + y_3y_5 + y_2y_6), (y_2, y_3y_4^2 + y_3^2y_5 + y_1y_5^2 + y_1y_4y_6),$$
$$(y_3, y_2y_4^2y_5 + y_1y_4y_5^2 + y_1y_4^2y_6 + y_2^2y_5y_6 + y_1y_2y_6^2),$$
$$(y_4, y_2y_3^2y_5 + y_1y_2y_5^2 + y_2^2y_3y_6 + y_1y_3^2y_6 + y_1^2y_5y_6),$$
$$(y_5, y_3^2y_4 + y_1y_3y_6 + y_2y_4^2 + y_2^2y_6), (y_6, y_3^2 + y_2y_4 + y_1y_5)\}$$

Mutating at y_1 we obtain the seed

$$\{(y_7, y_4^2 + y_3y_5 + y_2y_6), (y_2, y_5^2 + y_4y_6 + y_3y_7),$$
$$(y_3, y_4y_5^2 + y_4^2y_6 + y_2y_6^2 + y_2y_5y_7),$$
$$(y_4, y_3y_5^2y_6 + y_2y_5y_6^2 + y_2y_5^2y_7 +, y_3^2y_6y_7 + y_2y_3y_7^2),$$
$$(y_5, y_3y_4^2y_6 + y_2y_3y_6^2 + y_3^2y_4y_7 + y_2y_4^2y_7 + y_2^2y_6y_7),$$
$$(y_6, y_4^2y_5 + y_2y_4y_7 + y_3y_5^2 + y_3^2y_7)\}$$

where y_7 is the new cluster variable, related to y_1 via the formula

$$y_1y_7 = y_4^2 + y_3y_5 + y_2y_6.$$

Gale-Robinson LP algebra

After mutating at $y_1, y_2, \ldots, y_k$ we will have a seed containing $y_{k+1}, y_{k+2}, \ldots, y_{k+6}$.

So the Laurent phenomenon for the Gale-Robinson sequence follows from the result for LP algebras.

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So the Laurent phenomenon for the Gale-Robinson sequence follows from the result for LP algebras.
But we can also mutate in other directions. Here's another seed:

$$\begin{gathered}
\{(v, uw^2y_5^4 + u^2y_6z + w^3y_5^2(y_5y_6+z)^2), \\
(w, u^3vy_6^5 + v^3z + u^5y_6^3(y_5y_6+z)), \\
(z, u + wy_6^2), \\
(u, v^2z^2 + vw^2y_5y_6^3z(y_5y_6+z) + w^5y_5^3(y_5y_6+z)^4), \\
(y_5, u^3vy_6^5 + u^2vwy_6^7 + v^3z + u^4wy_6^5z), \\
(y_6, u^2w^6y_5^{12} + 2uw^7y_5^{10}z^2 + z^2(v^2w^3y_5^5 + v^3z + w^8y_5^8z^2))\}
\end{gathered}$$

There's a lot of (unexplained) positivity going on...

Finite type cluster algebras

A cluster algebra is of finite type if it has finitely many clusters. One of the highlights of the theory of cluster algebras is the classification of finite type cluster algebras.

Theorem (Fomin-Zelevinsky)

Cluster algebras of finite type have the same classification as the Cartan-Killing classification of semisimple complex Lie algebras.

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Cluster algebras of finite type have the same classification as the Cartan-Killing classification of semisimple complex Lie algebras.

The cluster complex of a cluster algebra is the simplicial complex with base set the set of cluster variables, and with simplices given by subsets of cluster variables belonging to the same cluster.

Theorem (Fomin-Zelevinsky, Chapoton-Fomin-Zelevinsky)

The cluster complex of a finite type cluster algebra is the dual complex to a polytope called a generalized associahedron.

Generalized associahedra and Catalan combinatorics

In type A, the generalized associahedron is the usual associahedron. It has Catalan number of vertices, and is a pentagon in two dimensions.

In type B, the generalized associahedron is the cyclohedron. It is a hexagon in two dimensions.

Generalized associahedra give a uniform root-system theoretic way of developing Catalan-style combinatorics to all (finite) root systems.

Classifying finite type LP algebras

In dimension 3 there are only two polytopes which come up as generalized associahedra for irreducible root systems: the (three-dimensional) associahedron and cyclohedron.

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There are lots more types of LP algebras than cluster algebras!

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A classification may to some extent be impossible, because as we shall explain it would include a classification of isomorphism types of directed graphs.

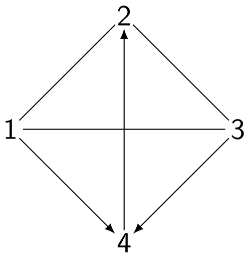
Linear LP algebras

Let us consider LP algebras $\mathcal{A}_\Gamma$ with a linear seed of the following form: $\{(X_i, F_i = A_i + \sum_{i\to j} X_j)\}$ where $i \to j$ are the edges in a fixed directed graph Γ. For example we might have the seed

$$\{(X_1, A_1 + X_2 + X_3 + X_4), (X_2, A_2 + X_1 + X_3),$$

$$(X_3, A_3 + X_1 + X_2 + X_4), (X_4, A_4 + X_2)\}.$$

associated to



Acyclic functions

Let $I \subset \{1, 2, \ldots, n\} = [n]$ be a set of vertices of Γ. A function $f : I \to [n]$ is acyclic if

1. For each $i \in I$, either $f(i) = i$, or $i \to f(i)$ is an edge of Γ; and
2. the subgraph consisting of the edges $i \to f(i)$ is has no directed cycles.

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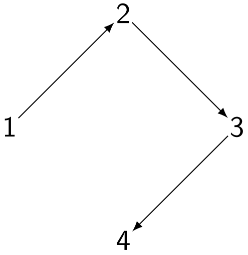
Suppose $I = \{1, 2, 3\}$.
Take $f(1) = 2$, $f(2) = 3$, $f(3) = 4$.



ACYCLIC

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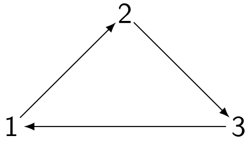
Suppose $I = \{1, 2, 3\}$.
Take $f(1) = 2,\ f(2) = 3,\ f(3) = 1$.



NOT ACYCLIC

Y-variables

For a subset I of vertices define

$$Y_I = \frac{\sum_{\text{acyclic } f\colon I \longrightarrow [n]} \prod_{i \in I} \tilde{X}_{f(i)}}{\prod_{i \in I} X_i}$$

where

$$\tilde{X}_{f(i)} = \begin{cases} X_{f(i)} & \text{if } i \neq f(i) \\ A_{f(i)} & \text{if } i = f(i). \end{cases}$$

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$$Y_{124} = \frac{X_1(X_2(X_3 + A_1) + A_4(X_3 + X_4 + A_1))}{X_1 X_2 X_4} + \frac{(X_2 + A_4)(X_2 + X_3 + X_4 + A_1)(X_3 + A_2)}{X_1 X_2 X_4}$$

Nested complex

Let $\mathcal{I} \subset 2^{[n]}$ denote the collection of strongly-connected subsets of Γ. A family of subsets $\mathcal{S} = \{I_1, \ldots, I_k\} \in \mathcal{I}$ is nested if

- for any pair I_i, I_j either one of them lies inside the other, or they are disjoint;
- for any tuple of disjoint I_j-s, they are the strongly connected components of their union.

The support S of a nested family $\mathcal{S} = \{I_1, \ldots, I_k\}$ is $S = \bigcup I_j$. A nested family is maximal if it is not properly contained in another nested family with the same support.

Linear LP algebras

Theorem (L.-Pylyavskyy)

- *The cluster variables of* $\mathcal{A}_\Gamma$ *are exactly* $X_1, X_2, \ldots, X_n$ *and* Y_I *for* $I \subset [n]$ *a strongly connected subset of* Γ*.*
- *The clusters of* $\mathcal{A}_\Gamma$ *are in bijection with the maximal nested families* $\mathcal{S} = \{I_1, \ldots, I_k\}$ *of* Γ*:*

$$\{X_i \mid i \notin I_1 \cup I_2 \cup \cdots \cup I_k\} \cup \{Y_{I_1}, Y_{I_2}, \ldots, Y_{I_k}\}$$

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- *The cluster variables of $\mathcal{A}_\Gamma$ are exactly $X_1, X_2, \ldots, X_n$ and Y_I for $I \subset [n]$ a strongly connected subset of Γ.*
- *The clusters of $\mathcal{A}_\Gamma$ are in bijection with the maximal nested families $\mathcal{S} = \{I_1, \ldots, I_k\}$ of Γ:*

$$\{X_i \mid i \notin I_1 \cup I_2 \cup \cdots \cup I_k\} \cup \{Y_{I_1}, Y_{I_2}, \ldots, Y_{I_k}\}$$

One of the clusters in our example is $\{Y_3, Y_{23}, Y_{123}, X_4\}$. The cluster variable Y_1 belongs to another cluster, and has the following Laurent polynomial expression

$$Y_1 = \frac{1 + Y_3^2 + Y_{23} + Y_3(2 + Y_{123})}{Y_3 Y_{23}}.$$

Note that this expression is also positive, which we don't yet have a general explanation for even in this linear LP case.

Nested complexes as cluster complexes

1. We don't know if the cluster complex of $\mathcal{A}_\Gamma$ is dual the face complex of a polytope.
2. But inside it is a subcomplex called the nested complex studied by Feichtner and Sturmfels, and by Postnikov. This subcomplex is the cluster complex for some "frozen" modification of $\mathcal{A}_\Gamma$.
3. The nested complex is dual to a polytope called the nestohedron, which includes a class of polytopes known as graph associahedra, a name coined by Carr and Devadoss, and also studied by De Concini and Procesi, and Toledano-Laredo.

Graph associahedra and nestohedra

Zelevinsky: noted "striking similarity" between nested complexes and cluster complexes/generalized associahedra.
Cluster complexes of finite type LP algebras are a common generalization of nestohedra and generalized associahedra.

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For a subset $I \subset \{1, 2, \ldots, n\}$ let

$$\Delta_I = \text{convex hull}(e_i \mid i \in I) \subset \mathbb{R}^n$$

Then the digraph associahedron $P(\Gamma)$ is given by the Minkowski sum

$$P(\Gamma) = \sum_I \Delta_I$$

where the sum is over all strongly connected subsets of Γ.

Total positivity

One of the main initial examples of cluster algebras are those associated to double Bruhat cells of seimsimple Lie groups. For double Bruhat cells of GL_n, these cluster algebras encode combinatorics related to wiring diagrams, planar networks, and total positivity.

A real matrix is totally positive if all minors of the matrix are strictly positive.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$a > 0, b > 0, \ldots \qquad ae - bd > 0, ai - cg > 0, \ldots \qquad det > 0$

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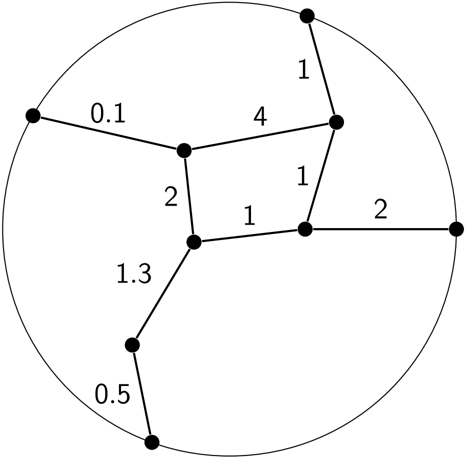
Question

How many minors do you need to check to ensure that a square $n \times n$ matrix is totally positive?

Turns out that some such collections of minors form clusters in the cluster algebra of the open double Bruhat cell of GL_n.

Electrical networks

An electrical network consisting only of resistors can be modeled by an undirected weighted graph Γ.



Response matrices

The response matrix

$$L(\Gamma) : \mathbb{R}^{\#\text{boundary vertices}} \to \mathbb{R}^{\#\text{boundary vertices}}$$

describes the current that flows through the boundary vertices when specified voltages are applied.

Inverse problem

To what extent can we recover Γ from $L(\Gamma)$?

Detection problem

Given a matrix M, how can we tell if $M = L(\Gamma)$ for some Γ?

Connection problem

Given $L(\Gamma)$, how many (algebraic) functions do we need to check to ensure that Γ is well-connected? (Is as well-connected as any planar electrical network?)

Planar electrical networks

For a planar electrical network, these problems were studied (and to a large extent solved) by Curtis-Ingerman-Morrow and de Verdière-Gitler-Vertigan.

Electrical Lie groups

To study electrical networks, we introduced an electrical Lie group EL_{2n} acting on electrical networks.

$$\begin{aligned} GL_n &\leftrightarrow EL_{2n} \\ \text{totally positive matrices} &\leftrightarrow \text{response matrices} \\ \text{minors} &\leftrightarrow \text{electrical measurements} \\ \text{planar directed networks} &\leftrightarrow \text{electrical networks} \\ \text{Serre relation}[e,[e,e']]=0 &\leftrightarrow \text{electrical Serre relation}[e,[e,e']]=-2e \end{aligned}$$

Electrical cells as Laurent phenomenon algebras

Theorem (Berenstein-Fomin-Zelevinsky, Geiss-Leclerc-Schröer)

The uni-upper triangular subgroup $U_n \subset GL_n$ has a Bruhat decomposition

$$U_n = \bigsqcup_{w \in S_n} C_w$$

such that the coordinate ring $\mathbb{C}[C_w]$ is naturally equipped with the structure of a cluster algebra.

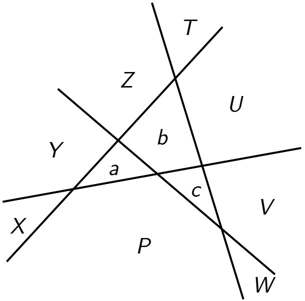
Conjecture

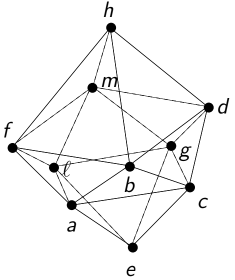
The electrical Lie group EL_{2n} has a decomposition

$$EL_{2n} = \bigsqcup_{w \in S_{2n}} A_w$$

such that each coordinate ring $\mathbb{C}[A_w]$ is naturally equipped with the structure of a Laurent phenomenon algebra.

Electrical LP algebras

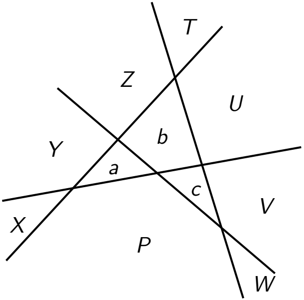


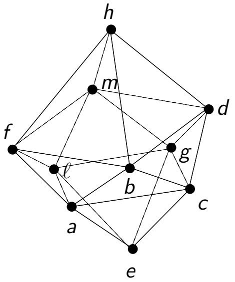


$$\{(a, bX+cY+PZ), (b, acT+cUY+PUZ+aVZ), (c, PU+aV+bW)\}$$

The rule for the exchange polynomials appeared in work of Henriques and Speyer, but also can be deduced from our general theory.

Electrical LP algebras





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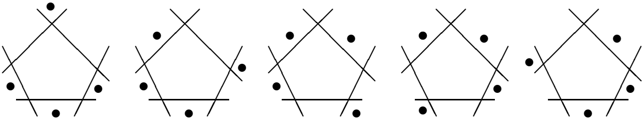

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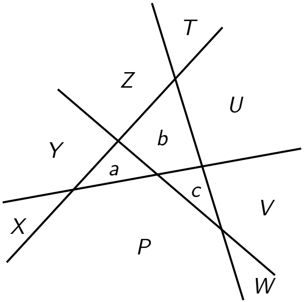

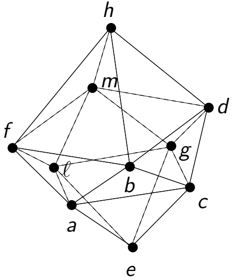

$$\{(a, bX+cY+PZ), (b, acT+cUY+PUZ+aVZ), (c, PU+aV+bW)\}$$

The exchange relations for a and c

$$ad = bX + cY + PZ \qquad \text{and} \qquad cf = PU + aV + bW.$$

are instances of the cube recurrence studied by Propp, Carroll, Speyer, Henriques, Fomin, Zelevinsky...

Another seed is

$$\{(a, e + UX), (e, acT + cUY + PUZ + aVZ), (c, e + WZ)\}$$

which does not come from a wiring diagram ("non-Plucker seed"). Each of the 16 seeds looks like either this seed, or the initial seed.

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$$\{(a, e + UX), (e, acT + cUY + PUZ + aVZ), (c, e + WZ)\}$$

which does not come from a wiring diagram ("non-Plucker seed"). Each of the 16 seeds looks like either this seed, or the initial seed.

The cluster variable m is given by the Laurent polynomial

$$\frac{1}{abc}(acPT + bPUX + abVX + b^2WX + cPUY + \\ bcWY + P^2UZ + aPVZ + bPWZ)$$

with respect to the initial seed.

The end

Thankyou!