Name:

Final Exam for Math 131, Fall 2008-2009.

Saturday January 17, 2009.
Time allowed: 3 hours.

Instructions.

1. Answer all parts of all questions on the exam.
2. Write your answers on this exam paper. You may use the back of each page. Ask for more paper if you need it.
3. No calculators or books allowed.
4. You can use any results proved in the lectures or in the textbook. You may not quote results proved in the problem sets, unless you prove them again.
5. Apart from the first question, all answers must be justified.
6. Full score is 100 points.

1	/10
2	/15
3	/18
4	/8
5	/6
6	/6
7	/7
8	/10
9	/6
10	/6
11	/8
12	/5
Total:	/105

1. (10 points)

   Mark each statement true (T) or false (F).

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (a) The space X obtained from a hexagon with sides identified according to the labeling scheme $abca^{-1}bc$ is a surface.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (b) Suppose $r : X \to A$ is a retract of X onto a subspace $A \subset X$, and let $x_0 \in A$. Then the map $r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (c) Suppose $p : E \to B$ is a covering map, and $E' \subset E$ is a subspace. Then $E' \to p(E')$ is a covering map.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (d) Let X be a topological space. Then X is contractible (that is, the identity map $\mathrm{id} : X \to X$ is nullhomotopic) if and only if X deformation retracts to a point.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (e) A retract of a contractible space X is contractible.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (f) Suppose $p : E \to B$ is a covering map and $f : [0,1] \to B$ is continuous. Assume all spaces are path-connected. Then f lifts uniquely to $\tilde{f} : [0,1] \to E$.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (g) Suppose $p : E \to B$ is a covering map and $f : X \to B$ is continuous. Assume all spaces are path-connected. Suppose E is simply-connected but X is not simply-connected. Then no lift $\tilde{f} : X \to E$ of f exists.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (h) There is no retraction from $S^1 \times B^2$ to its boundary torus $S^1 \times S^1$.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (i) Let $X \subset \mathbb{R}^2$ be the union of the circles C_n with radius n and center $(n, 0)$, for $n = 1, 2, \ldots$. Then X is homotopy-equivalent to the countable wedge of circles.

$\boxed{\text{T}}$ $\boxed{\text{F}}$ (j) There is a covering map from the 3-fold torus to the 2-fold torus.

2. (15 points) What are the fundamental groups of the following spaces? (Give brief explanations.)

   (a) $S^1 \times S^2$.

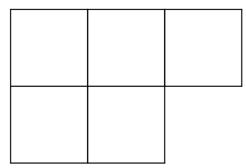
   (b) $S^1 \vee S^2$.

   (c) A two-fold torus.

   (d) The quotient space of S^3 obtained by identifying every point with its antipode.

   (e) The octagon with sides identified according to the labeling scheme $b^{-1}abbbba^{-1}b$. (Write the answer as a free product of two well-known groups.)

   (f) The wireframe



   with topology inherited as a subspace of $\mathbb{R}^2$. (Only the lines are in this space.)

(Extra space for Problem 2)

3. (18 points)

(a) Let X be a torus, and $p : E \to X$, $p' : E' \to X$ two covering maps of X. If E and E' are both two-fold (i.e. $|p^{-1}(x)| = 2 = |(p')^{-1}(x)|$ for $x \in X$) path-connected covering spaces, are they necessarily equivalent?

(b) (One-dimensional fixed point theorem?) Suppose $f : [0,1] \to [0,1]$ is continuous. Is there always an $x \in [0,1]$ such that $f(x) = x$?

(c) For which $m \geq 1$ is the fundamental group of the m-fold projective plane abelian? Prove it.

(d) Suppose $p : E \to B$ is a covering map, and E is not simply-connected. Assume both E and B are path-connected. Prove or disprove: B is not simply-connected.

(e) Is there a continuous map $f : X \to S^1$ with $\pi_1(X) = \mathbb{Z}/5\mathbb{Z}$ which is not null-homotopic?

(Extra space for Problem 3.)

4. (8 points) Consider the space X obtained from an octagon with sides identified according to the labeling scheme $ada^{-1}bcdcb$.

   Figure out which standard surface (n-fold torus or n-fold projective plane) X is homeomorphic with, using *explicit cut and paste operations*.

5. (6 points) Let $X = \{(x, y) \mid x \geq 0\} \subset \mathbb{R}^2$ be equipped with the subspace topology. Carefully prove that X is not a 2-manifold.

6. (6 points) Let $p : E \to B$ be a covering space such that $p^{-1}(b)$ is a finite set of points for each $b \in B$. Suppose B is compact. Prove that E is compact.

7. (7 points) Consider the topological space X constructed as follows: take two tori $Y_1 = S^1 \times S^1$ and $Y_2 = S^1 \times S^1$ and glue them (that is, construct the quotient space using the equivalence relation) using a homeomorphism between the two circles $S^1 \times \{1\} \subset Y_1$ and $S^1 \times \{1\} \subset Y_2$. Calculate, using the Seifert van-Kampen theorem, the fundamental group of X.

8. (10 points) Let $X = S_1 \vee S_1$.

(a) Draw a picture of the universal covering space of X (no explanation needed).

(b) Let a and b be the two standard generators of $\pi_1(X)$. Find and draw a 3-fold covering map $p : E \to X$ such that $p_*(\pi_1(E))$ contains both a^3 and b^3.

(c) What is the universal covering space of the projective plane P^2? What is the universal covering space of the wedge $P^2 \vee P^2$ of two projective planes?

9. (6 points) Let X be a path-connected space and $x_0 \in X$. Prove or give a counterexample to the following statement: if $f : X \to X$ is a continuous map satisfying $f(x_0) = x_0$, and f is homotopic to the identity map, then $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity map.

10. (6 points) Let X be a topological space, and let $F(x,t) : X \times [0,1] \to X$ be a homotopy, where $F(x,0) = F(x,1)$ is the identity map. Prove that for $x_0 \in X$, the loop $F(x_0,t)$ represents an element in the center of $\pi_1(X,x_0)$. (Recall that the center Z of a group G is the set $Z = \{z \in G \mid zg = gz \text{ for all } g \in G\}$.)

11. (8 points)

(a) Let X be a topological space. Suppose $F : [0,1]^2 \to X$ is a homotopy between two paths $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$, that is $F(x,0) = \alpha(x)$ and $F(x,1) = \beta(x)$.

Let $\mathcal{C}([0,1], X)$ be the space of continuous functions from [0,1] to X, equipped with the compact-open topology. Prove that the function $f : [0,1] \to \mathcal{C}([0,1], X)$ given by

$$t \longmapsto (x \mapsto F(x,t))$$

is continuous.

(Recall that the compact-open topology of $\mathcal{C}(Y, Z)$ has subbasis consisting of the sets

$$S(A, U) = \{f \in \mathcal{C}(Y, Z) \mid f(A) \subset U\}$$

where $A \subset Y$ is compact and $U \subset Z$ is open.)

(b) When does the converse of (a) hold? That is, if f is continuous, does that imply that F is continuous?

(c) What are the path-components of $\mathcal{C}([0,1], S^1)$?

(d) What are the path-components of $\mathcal{C}(S^1, S^1)$?

12. (5 bonus points) *Warning: This problem is hard and the points may not justify the time.*

    (a) Let S^1 be embedded in $\mathbb{R}^3$ in a boring fashion (as the boundary of an embedded disk). The space $\mathbb{R}^3 - S^1$ deformation retracts to the wedge $S^1 \vee X$ of a circle and a familiar space X. What is this familiar space X? Prove it.

    (b) Let $A \subset \mathbb{R}^3$ be the disjoint union of two circles embedded in $\mathbb{R}^3$ in an unlinked and unknotted manner. Let $B \subset \mathbb{R}^3$ be the disjoint union of two circles embedded in $\mathbb{R}^3$ linked, but otherwise not knotted. Calculate $\pi_1(\mathbb{R}^3 - A)$ and $\pi_1(\mathbb{R}^3 - B)$ and deduce that $\mathbb{R}^3 - A$ and $\mathbb{R}^3 - B$ are not homotopy-equivalent.