Math 131 Topology. Final Exam. May 23, 2002

When you show some property or make some construction, you can be very brief. Just remember to indicate clearly the important elements that enter into your argument.

1. (40 points) A topological space X is said to be **Hausdorff** if any two distinct points of X have disjoint open neighborhoods. Show that X is Hausdorff if and only if the **diagonal** $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ is closed in $X \times X$.

2. (40 points) A topological space X is said to be **locally path-connected at x** if for every neighborhood U of x there is a path-connected neighborhood of x contained in U. Show that X is locally path-connected if and only if for every open subset U of X each path-component of U is open in X. If X is locally path-connected, is it true that every connected open set in X is path-connected?

3. (40 points) Is there a topological space that is totally disconnected, uncountable, and compact? If so, exhibit it and establish its properties; if not, prove not.

4. (40 points) Let $p : X \to Y$ be a quotient map with the property that Y is connected and all of the fibers $p^{-1}(y) \subset X$ are connected (all $y \in Y$). Is it true that X is then connected? If so prove it; if not find a counter-example.

5. (40 points) Let X be a metric space (with metric denoted by $d(x, y)$) and A and B two disjoint closed subspaces of X. Without invoking the Urysohn Lemma construct a continuous real-valued function on X which takes the constant value 0 on A and 1 on B. Hint: recall the continuous functions $d(x, A)$ and $d(x, B)$ which measure the distance from a point $x \in X$ to the sets A and B respectively; but if you do so, define these functions carefully and prove that that they are indeed continuous.

6. (40 points) Let $p : E \to B$ be a covering map, with E (nonempty, and) path-connected, and B path-connected and simply connected. Must p be a homeomorphism? If so prove this; if not, prove not.

7. (40 points) Given that the fundamental group of the circle is not the trivial group, sketch the proof of the theorem that says that any continuous mapping of the closed unit disc

$$\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$$

into itself has a fixed point (i.e., the *Brouwer fixed point theorem for the disc*).

8. (20 points) Let $Z \subset \mathbf{R}^3$ be the "vertical line"

$$Z = \{(0, 0, z) \in \mathbf{R}^3 \text{ for } z \in \mathbf{R}\}.$$

What is the fundamental group of the complement of Z in $\mathbf{R}^3$?