Math 131 Final Exam

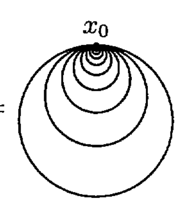
Wednesday, May 26, 2004

1. *(15 points)* A *proper* map $f : X \to Y$ is, by definition, a continuous map where the inverse image of a compact set is compact. Let $X^* = X \cup \{\infty\}$ be the one-point compactification of X, and let Y^* be the one-point compactification of Y. For an arbitrary function $f : X \to Y$, define $f^* : X^* \to Y^*$ by

$$f^*(x) = \begin{cases} \infty & x = \infty \\ f(x) & \text{otherwise.} \end{cases}$$

Show that f^* is continuous if and only if f is continuous and proper. Conclude that a proper map to a locally compact Hausdorff space is a closed map.

2. Let H be the Hawaiian earring

$H =$ ,

the union over $n \in \mathbb{N}$ of circles C_n of radius r^n centered at $(0, -r^n)$ (with $r < 1$). Let W_∞ be the graph with one vertex x_0 and a countable infinite set of edges $E = \{e_n \mid n \in \mathbb{N}\}$. Recall that this is the quotient space of an infinite disjoint union of intervals (one for each n) where all endpoints are identified. (W_∞ is also known as the wedge of an infinite number of circles.)

   (a) *(10 points)* Show that W_∞ is normal but is not first countable and so is not a metric space.

   (b) *(20 points)* Consider the loops γ_n in H (resp. W_∞) which run clockwise around C_n (resp. along e_n from start to end). In which topologies on $C([0,1], H)$ (resp. $C([0,1], W_\infty)$) does the sequence of paths

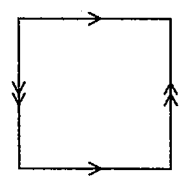
$$f_n = \gamma_1 * (\gamma_2 * (\gamma_3 * \cdots * (\gamma_{n-2} * (\gamma_{n-1} * \gamma_n)) \cdots))$$

converge? When does the sequence

$$g_n = ((\cdots((\gamma_1 * \gamma_2) * \gamma_3) * \cdots * \gamma_{n-2}) * \gamma_{n-1}) * \gamma_n$$

converge? Consider the product, uniform, compact convergence, and compact open topologies.

3. Recall that the Klein bottle K is obtained by taking the quotient space of the square



with the equivalence relation where opposite edges are identified ("glued") according to the arrows above.

   (a) *(10 points)* Find an explicit covering map from $\mathbb{R}^2$ to K. (Hint: Think about how we showed that $\mathbb{R}^2$ is the universal covering space of the torus.)

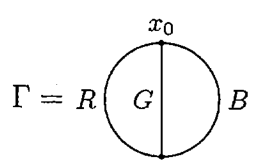
   (b) *(10 points)* Use the map you found above to find the fundamental group of the Klein bottle. Explicitly specify the multiplication and inverse.

   (c) *(10 points)* Use the Seifert-van Kampen theorem to give another description of $\pi_1(K)$.

4. The *degree* of a covering map $p : E \to B$ at a point $x \in B$ is the number of elements in $p^{-1}(\{x\})$; it may be a finite number or ∞.

   (a) *(10 points)* Show that the degree is a continuous map from B to $\mathbb{N} \cup \infty$ in the discrete topology. Conclude that if B is connected the degree is constant, so we may talk about the degree of the covering space.

   (b) *(15 points)* Find all covering spaces of degree 2 of the theta graph



   For each one, find the corresponding set action of $\pi_1(\Gamma, x_0)$ and (if the covering space is connected) generators for the corresponding subgroup of $\pi_1(\Gamma, x_0)$, using the classification of covering spaces we developed.