Some extra problems

Many of the problems are "True or False" questions. (That usually makes them harder.) If true, you should prove the statement. If false, you should provide a counterexample.

Some of the later questions are quite difficult and should be considered challenge problems.

1. Let A and B be square matrices of the same size. True or False:
$$(A+B)(A-B) = A^2 - B^2.$$

2. True or False: there is a 3×3 matrix A so that $A^7 = I$, but $A, A^2, \ldots, A^6$ are all different from I.

3. Let A be a square matrix such that the sum of the entries in each row adds up to 1. What can you say about the matrix $I - A$?

4. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Suppose
$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{x}\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$
Prove that $\mathbf{x} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

5. Bob thinks that
$$A = \begin{bmatrix} -3 & 1 & 2 \\ -1 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$$
is the square $A = B^2$ of another matrix B. Do you believe him?

   Alice thinks that the same matrix A can be written as $A = CC^T$ for some (possibly non-square) matrix C. Do you believe her?

6. Let A be the same matrix as in the previous problem. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are linearly independent vectors. True or False: it is always the case that $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$ are linearly independent.

7. Take the same matrix A again. Change at most one number in A to obtain a matrix B, and find a vector $\mathbf{v} \in \mathbb{R}^3$, so that $B\mathbf{x} = \mathbf{v}$ has no solution. *This is the kind of thing I find myself doing a lot before an exam or quiz...*

8. Let A and B be any two matrices so that the transformation
$$\mathbf{x} \mapsto AB\mathbf{x}$$
is one-to-one. True or False: $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. (Also, same question but for B.)

9. Let A, B, C be 2×2 matrices, and suppose that A and C are invertible. Show that the 4×4 matrix (composed of 2×2 blocks)

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

is also invertible, and write down its inverse.

10. Prove that given any five vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5 \in \mathbb{R}^3$, one can find numbers $c_1, c_2, \ldots, c_5$ such that both

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_5\mathbf{v}_5 = \mathbf{0} \qquad \text{and} \qquad c_1 + c_2 + \cdots + c_5 = 0.$$

11. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Let $S : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation with standard matrix A^T. Prove or disprove: T is one-to-one if and only if S is onto.

12. The trace $tr(A)$ of a square $n \times n$ matrix A is the sum $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$ of diagonal entries. Prove that $tr(AB) = tr(BA)$ for two $n \times n$ matrices A and B.

13. Let A be a square matrix. We know (do we?) that A is invertible if and only if we can find a square matrix B such that $AB = I$.

    True or False: A is singular if and only if we can find a square matrix B such that $AB = 0$.

14. Let a, b, c be numbers. Show that the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

is $-(a-b)(b-c)(a-c)$. Generalize this to $n \times n$ matrices. Can you prove it?

15. Let A be the 2×2 matrix corresponding to a rotation by $45°$. Let B be the 2×2 matrix corresponding to reflection in the x-axis. How many different matrices can you get by multiplying a bunch of A's and a bunch of B's in some order? For example, matrices such as

$$ABBA, \qquad \text{or} \qquad AAABABA.$$

16. Suppose that $A = -A^T$ (such a matrix is called *skew-symmetric*). Show that $I + A$ is invertible.

17. Let A and B be 2×2 matrices. Is it always possible to find numbers a, b not both zero, so that $\det(aA + bB) = 0$?