Some extra problems

1. Let $\langle ., . \rangle_1$ and $\langle ., . \rangle_2$ be two inner products on a vector space V. Is $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle_1 + \langle \mathbf{v}, \mathbf{w} \rangle_2$ an inner product on V?

2. Is there an inner product on $\mathbb{R}^2$ given by

$$\langle \mathbf{v}, \mathbf{w} \rangle = av_1w_1 + bv_1w_2 + cv_2w_1 + dv_2w_2$$

where a, b, c, d are all non-zero real numbers?

3. Let $V = \mathbb{P}_2$ denote the vector space of real polynomials with degree less than or equal to 2. Is

$$\langle p(t), q(t) \rangle = p(1)q(1) + p'(1)q'(1) + p''(1)q''(1)$$

an inner product on V?

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Suppose that $\det(A) = 0$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for $\mathbb{R}^n$. Prove that the matrix $[T]_\mathcal{B}$ of T relative to $\mathcal{B}$ satisfies $\det([T]_\mathcal{B}) = 0$.

5. Let A be a $m \times n$ matrix. Prove that every vector in the row space of A is orthogonal to every vector in the nullspace of A.

6. Does there exist a 3×3 matrix whose eigenvalues include $2 - i$ and $i + 1$?

7. Let V be an inner product space. Suppose you are told that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of V. Explain how you can determine $\langle \mathbf{u}, \mathbf{w} \rangle$ for eveyr $\mathbf{u}, \mathbf{w} \in V$.

8. Find similar matrices A and B with different eigenvectors.

9. Let $V = M_{2\times 2}$ be the set of 2×2 matrices with real entries. Check that V is a vector space when equipped with matrix addition, and scalar multiplication of matrices. Which of the following subsets form subspaces?

   (a) The set of invertible 2×2 matrices.

   (b) The set of singular 2×2 matrices.

   (c) The set of symmetric matrices, satisfying $A^T = A$.

   (d) The set of skew-symmetric matrices, satisfying $A^T = -A$.

   (e) The set of trace zero matrices, satisfying $\operatorname{tr}(A) = 0$.

   (f) The set of matrices A for which $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has a solution.

10. Let V be an inner product space and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Form the $n \times n$ matrix $A = (a_{ij})$ where $a_{ij} = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. Prove that A is invertible.

11. Suppose A is a square matrix of rank 1. Prove that there exist vectors $\mathbf{v}, \mathbf{w}$ so that $A = \mathbf{v}\mathbf{w}^T$.

12. Suppose A is a square matrix of rank 1. Prove that

$$\det(A + I) = \operatorname{tr}(A) + 1.$$

13. Let A and B be 5×5 matrices such that $AB = 0$. What are the possible ranks of A and B?

14. Let A be a 2×2 matrix. Prove or provide a counterexample: there are eigenvectors $\mathbf{v}, \mathbf{w}$ of A satisfying $\mathbf{v}.\mathbf{w} = 0$.

15. Let $W \subset \mathbb{R}^n$ be a subspace. Prove that $W \cap W^\perp = \{0\}$.

16. Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Prove that

    (a) $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$

    (b) $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$

17. Let A be an $n \times n$ real symmetric matrix; i.e., all entries of A are real numbers and $A^T = A$. Prove that the eigenvalues of A are real.

18. Suppose V_1, V_2, V_3 are mutually orthogonal subspaces of $\mathbb{R}^n$. That is, for $i \neq j$, we have $\mathbf{v}.\mathbf{w} = 0$ for $\mathbf{v} \in V_i$ and $\mathbf{w} \in V_j$. Prove that

$$\dim(V_1) + \dim(V_2) + \dim(V_3) \leq n.$$

19. Prove that $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ for any three real numbers a, b, c.

20. Let A and B be $n \times n$ matrices satisfying

$$A + B = I \qquad \text{and} \qquad \operatorname{rank}(A) + \operatorname{rank}(B) = n.$$

    (a) Prove that $\operatorname{Col}(A) \cap \operatorname{Col}(B) = \{0\}$.

    (b) Prove that $A^2 = A$, $B^2 = B$, and $AB = BA = 0$.