Book Homework #13 Answers

Math 217 W11

6.2.6. Not orthogonal. (The second and third vectors are not orthogonal.)

6.2.10. Compute $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 0$. Since the $\boldsymbol{u}_i$ are nonzero and orthogonal, they are independent. Since there are three of them, they form an orthogonal basis of $\mathbb{R}^3$.

$$\boldsymbol{x} = \left(\frac{\boldsymbol{x}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\right)\boldsymbol{u}_1 + \left(\frac{\boldsymbol{x}\cdot\boldsymbol{u}_2}{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}\right)\boldsymbol{u}_2 + \left(\frac{\boldsymbol{x}\cdot\boldsymbol{u}_3}{\boldsymbol{u}_3\cdot\boldsymbol{u}_3}\right)\boldsymbol{u}_3 = \frac{4}{3}\boldsymbol{u}_1 + \frac{1}{3}\boldsymbol{u}_2 + \frac{1}{3}\boldsymbol{u}_3$$

6.2.16. $\text{proj}_{\boldsymbol{u}}\boldsymbol{y} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$; $\boldsymbol{y} - \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$; $\left\|\begin{pmatrix} -6 \\ 3 \end{pmatrix}\right\| = \sqrt{45}$. The distance from $\boldsymbol{y}$ to the line spanned by $\boldsymbol{u}$ is $\sqrt{45}$.

6.2.24.

a) True. (Orthogonal sets can include zero; but orthogonal sets not containing zero are always independent.)

b) False. (This is the definition of *orthogonal*.)

c) True. (Theorem 7 and definition of lengths.)

d) True. (Rescaling $\boldsymbol{v}$ by c will rescale $\boldsymbol{y}\cdot\boldsymbol{v}$ and $\boldsymbol{v}$ each by c, but also rescale the denominator $\boldsymbol{v}\cdot\boldsymbol{v}$ by c^2 to compensate.)

e) True. (The definition tells you how to compute the inverse.)

6.2.26. Theorem 4 tells us that the n orthogonal vectors are independent. Since they span W, they form a basis of W. Thus W is an n-dimensional subspace of $\mathbb{R}^n$, so it is all of $\mathbb{R}^n$.

6.3.6. First note that $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0$. Then use Thm 8, eqn 2 to get $\hat{\boldsymbol{y}} = \frac{-3}{2}\boldsymbol{u}_1 + \frac{5}{2}\boldsymbol{u}_2 = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$. That is, $\boldsymbol{y} \in \text{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

6.3.10. $\boldsymbol{y} = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$

6.3.14. $\begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{pmatrix}$

6.3.22.

a) True. (Proof of Orthogonal Decomposition Theorem.)

b) True. (The geometric interpretation of projection.)

c) True. (uniqueness statement embedded in Theorem 8)

d) False. (the best approximation is $\text{proj}_W\boldsymbol{y}$)

e) False. (but true in the special case $n = p$)

6.3.24.

a) By hypothesis, the $\boldsymbol{w}_i$ are pairwise disjoint, as are the $\boldsymbol{v}_j$. Also, $\boldsymbol{w}_i \cdot \boldsymbol{v}_j = 0$ for all i, j because $\boldsymbol{w}_i \in W$ and $\boldsymbol{v}_j \in W^{\perp}$.

b) For any $\boldsymbol{y} \in \mathbb{R}^n$, write $\boldsymbol{y} = \hat{\boldsymbol{y}} + \boldsymbol{z}$ for some $\hat{\boldsymbol{y}} \in W$ and $\boldsymbol{z} \in W^{\perp}$, as in the Orthogonal Decomposition Theorem. Then there exist scalars c_i, d_j such that $\boldsymbol{y} = \hat{\boldsymbol{y}} + \boldsymbol{z} = c_1\boldsymbol{w}_1 + \cdots + c_p\boldsymbol{w}_p + d_1\boldsymbol{v}_1 + \cdots d_q\boldsymbol{v}_q$. Thus the set $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_p, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q\}$ spans $\mathbb{R}^n$.

c) The set $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_p, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_q\}$ spans $\mathbb{R}^n$ by part (b) and is independent by part (a) and Theorem 4, so it is a basis of $\mathbb{R}^n$.

$$n = \dim \mathbb{R}^n = p + q = \dim W + \dim W^{\perp}$$

6.4.12. $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ (other answers are possible, but this is the only one that arises from the "standard" method)

6.4.14. $R = \begin{pmatrix} 7 & 7 \\ 0 & 7 \end{pmatrix}$

6.4.16. $Q = \begin{pmatrix} 1/2 & -1/\sqrt{8} & 1/2 \\ -1/2 & 1/\sqrt{8} & 1/2 \\ 0 & 2/\sqrt{8} & 0 \\ 1/2 & 1/\sqrt{8} & -1/2 \\ 1/2 & 1/\sqrt{8} & 1/2 \end{pmatrix}, R = \begin{pmatrix} 2 & 8 & 7 \\ 0 & \sqrt{8} & 12/\sqrt{8} \\ 0 & 0 & 6 \end{pmatrix}$

6.4.18.

a) False. (The three orthogonal vectors must be *nonzero* to be a basis for a three-dimensional subspace.)

b) True. (If $\boldsymbol{x} \notin W$, then $\boldsymbol{x} \neq \mathrm{proj}_W \boldsymbol{x}$.)

c) True. (Theorem 12.)

6.4.20. If $\boldsymbol{y} \in \mathrm{Col}\, A$, then there exists an $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{y}$. Then $\boldsymbol{y} = QR\boldsymbol{x} = Q(R\boldsymbol{x})$, so that $y \in \mathrm{Col}\, Q$.

Conversely, suppose $\boldsymbol{y} \in \mathrm{Col}\, Q$, so that there exists an $\boldsymbol{x}$ such that $\boldsymbol{y} = Q\boldsymbol{x}$ for some $\boldsymbol{x}$. Since R is invertible, the equation $A = QR$ can be rewritten in the form $Q = AR^{-1}$. This gives $\boldsymbol{y} = AR^{-1}\boldsymbol{x} = A(R^{-1}\boldsymbol{x})$, so that $\boldsymbol{y} \in \mathrm{Col}\, A$.

Combining these, $\mathrm{Col}\, A = \mathrm{Col}\, Q$.