# Book Homework #14 Answers

## Math 217 W11

**6.5.12.**

a) $\hat{\boldsymbol{b}} = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 6 \end{pmatrix}$

b) $\hat{\boldsymbol{x}} = \begin{pmatrix} 1/3 \\ 14/3 \\ -5/3 \end{pmatrix}$

**6.5.16.** $\hat{\boldsymbol{x}} = \begin{pmatrix} 2.9 \\ .9 \end{pmatrix}$

**6.5.18.**

a) True. (Paragraph following def'n of least-squares solution)

b) False. (Figure 1 and the preceding discussion)

c) True. (Equation (1) and following discussion)

d) False. (This formula only applies when the columns of $A$ are linearly independent.)

e) False. (Discussion following Example 4)

f) False. ("Numerical Note")

**6.5.20.** Suppose that $A\boldsymbol{x} = \boldsymbol{0}$. Then $A^TA\boldsymbol{x} = A^T\boldsymbol{0} = \boldsymbol{0}$. Since $A^TA$ is invertible, by hypothesis, $\boldsymbol{x} = \boldsymbol{0}$. Hence the columns of $A$ are linearly independent.

**6.5.22.** $A^TA$ has $n$ columns. Then rank $A^TA = n -$ dim Nul $A^TA = n -$ dim Nul $A =$ rank $A$, where the first and last equalities are by the rank-nullity theorem and the middle one uses Exercise 19 (which is odd and has a solution in the book).

**6.7.9.**

a) $\hat{p_2}(t) = 5$

b) $(p_2 - \hat{p_2})(t) = t^2 - 5$ completes an orthogonal basis. The multiple $q(t) = \frac{1}{4}(t^2 - 5)$ is correctly normalized.

**6.7.10.**

| Polynomial | $p_0$ | $p_1$ | $q$ | $p(t) = t^3$ |
|---|---|---|---|---|
| Values | $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} -27 \\ -1 \\ 1 \\ 27 \end{pmatrix}$ |

$\hat{p}(t) = \frac{0}{4}p_0(t) + \frac{164}{20}p_1(t) + \frac{0}{4}q(t) = \frac{41}{5}t$

**6.7.14.** We check the defining properties of an inner product in turn.

1.

$$\begin{aligned} \langle \boldsymbol{u}, \boldsymbol{v} \rangle &= T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) && \text{(definition of } \langle \bullet, \bullet \rangle) \\ &= T(\boldsymbol{v}) \cdot T(\boldsymbol{u}) && \text{(commutativity of dot product)} \\ &= \langle \boldsymbol{v}, \boldsymbol{u} \rangle && \text{(definition of } \langle \bullet, \bullet \rangle) \end{aligned}$$

2.

$$\begin{aligned}
\langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{w}\rangle &= T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{w}) \qquad \text{(definition of } \langle \bullet, \bullet \rangle) \\
&= (T(\boldsymbol{u})+T(\boldsymbol{v})) \cdot T(\boldsymbol{w}) \qquad \text{(linearity of } T) \\
&= T(\boldsymbol{u}) \cdot T(\boldsymbol{w}) + T(\boldsymbol{v}) \cdot T(\boldsymbol{w}) \qquad \text{(dot product distributes over addition)} \\
&= \langle \boldsymbol{u}, \boldsymbol{w}\rangle + \langle \boldsymbol{v}, \boldsymbol{w}\rangle \qquad \text{(definition of } \langle \bullet, \bullet \rangle)
\end{aligned}$$

3.

$$\begin{aligned}
\langle c\boldsymbol{u}, \boldsymbol{v}\rangle &= T(c\boldsymbol{u}) \cdot T(\boldsymbol{v}) \qquad \text{(definition of } \langle \bullet, \bullet \rangle) \\
&= (cT(\boldsymbol{u})) \cdot T(\boldsymbol{v}) \qquad \text{(linearity of } T) \\
&= c(T(\boldsymbol{u}) \cdot T(\boldsymbol{v})) \qquad \text{(bilinearity of dot product)} \\
&= c\langle \boldsymbol{u}, \boldsymbol{v}\rangle \qquad \text{(definition of } \langle \bullet, \bullet \rangle)
\end{aligned}$$

4. For each $\boldsymbol{u}$, we have $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = T(\boldsymbol{u}) \cdot T(\boldsymbol{u}) \geq 0$ (Theorem 1d). If $\boldsymbol{u} = \boldsymbol{0}$, then by linearity $T(\boldsymbol{u}) = \boldsymbol{0}$, and thus $\langle \boldsymbol{0}, \boldsymbol{0}\rangle = \boldsymbol{0} \cdot \boldsymbol{0} = 0$. Finally, if $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = 0$, then $T(\boldsymbol{u}) \cdot T(\boldsymbol{u}) = 0$, so $T(\boldsymbol{u}) = \boldsymbol{0}$ by Theorem 1d. Since $T$ is an isomorphism, this implies $\boldsymbol{u} = \boldsymbol{0}$.

   (Note that this part, and only this part, fails if we merely assume that $T$ is linear.)

**6.7.16.**

$$\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}\rangle - \langle \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{u}\rangle - \langle \boldsymbol{u}, \boldsymbol{v}\rangle - \langle \boldsymbol{v}, \boldsymbol{u}\rangle + \langle \boldsymbol{v}, \boldsymbol{v}\rangle \\
&= 1-0-0+1 \\
&= 2
\end{aligned}$$

Thus, $\|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{2}$.

**6.7.25.** $1, t, 3t^2-1$

**6.7.26.** $1, t, 3t^2-4$