Book Homework #4 Answers

Math 217 W11

2.1.20. The second column of AB will be $A\mathbf{0} = \mathbf{0}$.

2.1.22. Write $B = (\, \boldsymbol{b}_1 \;\; \boldsymbol{b}_2 \;\; \cdots \;\; \boldsymbol{b}_n \,)$, so that the columns of AB are $A\boldsymbol{b}_1, A\boldsymbol{b}_2, \ldots, A\boldsymbol{b}_n$. Since the $\boldsymbol{b}_i$ are linearly dependent, there exist scalars c_i, not all zero, such that $c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n = \mathbf{0}$. Then columns of AB are related by the same relation. (This uses linearity of matrix-vector multiplication.)

$$c_1(A\boldsymbol{b}_1) + \cdots + c_n(A\boldsymbol{b}_n) = A(c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n) = A\mathbf{0} = \mathbf{0}$$

2.1.24. For any $\boldsymbol{b} \in \mathbb{R}^m$, we have $A(D\boldsymbol{b}) = (AD)\boldsymbol{b} = I_m\boldsymbol{b} = \boldsymbol{b}$, so $D\boldsymbol{b}$ is a solution of $A\boldsymbol{x} = \boldsymbol{b}$. In particular, this system is always consistent. By Theorem 4, A has a pivot position in every row. Since there cannot be more than one pivot in any column, there are at least as many rows as columns.

2.1.28. We always have $\boldsymbol{u}^T\boldsymbol{v} = \boldsymbol{v}^T\boldsymbol{u}$. On the other hand, $\boldsymbol{u}\boldsymbol{v}^T$ and $\boldsymbol{v}\boldsymbol{u}^T$ are transposes of one another.

2.2.14. $B - C = (B - C)I = (B - C)(DD^{-1}) = ((B - C)D)\, D^{-1} = OD^{-1} = O$, so $B = C$.

2.2.16. $A = AI = A(BB^{-1}) = (AB)B^{-1}$, so A is invertible by Theorem 6a,b.

2.2.20.

a) B is invertible because $B = X(A - AX)^{-1}$ is a product of invertible matrices.

b) We continue to rearrange the matrix equation.

$$\begin{aligned} X &= B(A - AX) \\ X &= BA - BAX \\ X + BAX &= BA \\ (I + BA)X &= BA \end{aligned}$$

Now BA is invertible, since it is a product of invertible matrices (using part a). Now, by Exercise 16, $I + BA$ is invertible, and at last we have $X = (I + BA)^{-1}BA$.

2.2.22. If A is invertible, then Theorem 5 says that the system $A\boldsymbol{x} = \boldsymbol{b}$ is always consistent. By Theorem 4, the columns of A span $\mathbb{R}^4$.

2.2.38. $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. There is no such matrix C. If there were, then we'd have $CA = I$, giving $CA\boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^4$. On the other hand, the colums of A are linearly dependent, so there exist nonzero vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \mathbf{0}$, giving $C(A\boldsymbol{x}) = C\mathbf{0} = \mathbf{0} \neq \boldsymbol{x}$.

2.3.14. If the diagonal entries are all nonzero, then they can be used as pivots, and the matrix will be invertible.

If on the other hand there is a zero on the diagonal, then the columns must be linearly dependent (the column with a zero on the diagonal will be a linear combination of the columns further to the right), and the matrix cannot be invertible.

A lower-triangular matrix is invertible iff the diagonal entries are all nonzero.

2.3.20. By the boxed statement under the IMT, E, F are inverses to each other. Thus we have $EF = I = FE$. Thus E and F commute.

2.3.29. Since $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is not one-to-one, property (f) in the IMT fails. Thus, by IMT, A is not invertible, and property (i) fails also, so $\boldsymbol{x} \mapsto A\boldsymbol{x}$ does not maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Since A is not invertible, Theorem 2.9 implies that $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is not an invertible map.

2.3.30. Since $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is one-to-one, property (f) in the IMT holds. Thus, by IMT, A is invertible, and property (i) holds also, so $\boldsymbol{x} \mapsto A\boldsymbol{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Since A is invertible, Theorem 2.9 implies that $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is an invertible map.

2.3.34. The standard matrix of T is $A = \begin{pmatrix} 6 & -8 \\ -5 & 7 \end{pmatrix}$, which is invertible because $\det A = 2 \neq 0$. By Theorem 9, T is invertible, and $T^{-1}(\boldsymbol{x}) = A^{-1}\boldsymbol{x} = \frac{1}{2}\begin{pmatrix} 7 & 8 \\ 5 & 6 \end{pmatrix}\boldsymbol{x}$.

2.3.36. Suppose T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, and let A be the standard matrix of T. Then the columns of A span $\mathbb{R}^n$ by Theorem 1.12. By IMT, A is invertible. By Theorem 2.9, T is invertible, and A^{-1} is the standard matrix of T^{-1}. Since A^{-1} is also invertible, by the IMT, its columns are linearly independent and span $\mathbb{R}^n$. By Theorem 12 applied to T^{-1}, we see T^{-1} is one-to-one.

2.3.37. Let A be the standard matrix of T and let B be the standard matrix of U. Then for all $\boldsymbol{x} \in \mathbb{R}^n$, $AB\boldsymbol{x} = T(U(\boldsymbol{x})) = \boldsymbol{x}$. Since AB represents the identity transformation, $AB = I$. By the boxed statement following the IMT, $BA = I$, so $U(T(\boldsymbol{x}) = BA\boldsymbol{x} = \boldsymbol{x}$.