# Book Homework #6 Answers

## Math 217 W11

**4.1.13.**

a) No. There are three vectors in $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

b) There are infinitely many vectors in Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

c) Yes, $\boldsymbol{w} = \boldsymbol{v}_1 + \boldsymbol{v}_2$.

**4.1.14.** We can answer this systematically by row reducing $(\ \boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \boldsymbol{v}_3\ \ \boldsymbol{w}\ )$.

$$\begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This tells us that $\boldsymbol{w}$ is not a linear combination of the $\boldsymbol{v}_i$. (We can also see, if somehow we had not yet noticed, that $\boldsymbol{v}_3 = 2\boldsymbol{v}_2$.

**4.1.24.**

a) True.

b) True. (Blue box following vector space axioms, proof sketched in problem 29. Most importantly, this is not "by definition".)

c) Syntax error. It only makes sense to ask whether a vector space is a subspace of another vector space. Two true, well-formed sentences similar to the given one follow.

- A vector space is always a subspace of itself.
- A subspace of a vector space is always a vector space.

d) False. $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$.

e) False. Quantifier error. We need $\boldsymbol{u} + \boldsymbol{v} \in H$ for all $\boldsymbol{u}, \boldsymbol{v} \in H$, and we need $c\boldsymbol{u} \in H$ for all $c \in \mathbb{R}, \boldsymbol{u} \in H$.

**4.1.32.** We can check the three parts of the definition directly.

a) $\mathbf{0} \in H$ and $\mathbf{0} \in K$, so $\mathbf{0} \in H \cap K$

b) Suppose $\boldsymbol{u}, \boldsymbol{v} \in H \cap K$. Then, since $H$ is closed under addition, $\boldsymbol{u} + \boldsymbol{v} \in H$. Since $K$ is closed under addition, $\boldsymbol{u} + \boldsymbol{v} \in K$. Thus $\boldsymbol{u} + \boldsymbol{v} \in H \cap K$.

c) Suppose $\boldsymbol{u} \in H \cap K$ and $c \in \mathbb{R}$. Since $H$ is closed under scalar multiplication, $c\boldsymbol{u} \in H$. Since $K$ is closed under scalar multiplication, $c\boldsymbol{u} \in K$. Thus $c\boldsymbol{u} \in H \cap K$.

This shows that $H \cap K$ is a subspace of $V$.

Let $H$ be the $x$-axis in $\mathbb{R}^2$ and let $K$ be the $y$-axis, so that $H \cup K$ looks like a cross. This is not closed under addition. For example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in H \cup K$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in H \cup K$, but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin H \cup K$. (In general, the union of subspaces of $V$ is almost never a subspace of $V$. The only time $H \cup K$ is a subspace of $V$ is when one of $H, K$ contains the other, so that $H \cup K$ is just $H$ or $K$.)

**4.1.33.**

a) Since $\mathbf{0} \in H$ and $\mathbf{0} \in K$, $\mathbf{0} = \mathbf{0} + \mathbf{0} \in H + K$. Now, let $\boldsymbol{u} = \boldsymbol{h} + \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{h}' + \boldsymbol{k}'$ be arbitrary elements of $H + K$, and let $c \in \mathbb{R}$. Then $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{h} + \boldsymbol{h}') + (\boldsymbol{k} + \boldsymbol{k}') \in H + K$, and also $c\boldsymbol{u} = c\boldsymbol{h} + c\boldsymbol{k} \in H + K$.

b) We already know that $H$, $K$, and $H + K$ contain the zero vector in $V$ and are closed under linear combinations, so the only thing to check is that $H$ and $K$ are *subsets* of $H + K$. For all $\boldsymbol{h} \in H$, $\boldsymbol{h} = \boldsymbol{h} + \mathbf{0} \in H + K$. Likewise, for all $\boldsymbol{k} \in K$, we have $\boldsymbol{k} = \mathbf{0} + \boldsymbol{k} \in H + K$. Thus $H \subseteq H + K$ and $K \subseteq H + K$.

**4.1.34.** Let $\boldsymbol{u} = \boldsymbol{h} + \boldsymbol{k}$ be any element of $H + K$. Then we can write $\boldsymbol{h} = \sum_{i=1}^{p} c_i \boldsymbol{u}_i$ and $\boldsymbol{k} = \sum_{i=1}^{q} d_i \boldsymbol{v}_i$ for suitable scalars $c_i, d_i$. Then we have

$$\boldsymbol{u} = \boldsymbol{h} + \boldsymbol{k} = \sum_{i=1}^{p} c_i \boldsymbol{u}_i + \sum_{i=1}^{q} d_i \boldsymbol{v}_i \in \operatorname{Span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q\},$$

which gives $H + K \subseteq \operatorname{Span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q\}$.

For the other direction, let $\boldsymbol{u}$ be any element of $\operatorname{Span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q\}$. Then we can write

$$\boldsymbol{u} = \sum_{i=1}^{p} c_i \boldsymbol{u}_i + \sum_{i=1}^{q} d_i \boldsymbol{v}_i = \left( \sum_{i=1}^{p} c_i \boldsymbol{u}_i \right) + \left( \sum_{i=1}^{q} d_i \boldsymbol{v}_i \right) \in H + K,$$

so that $\operatorname{Span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q\} \subseteq H + K$.

Together, these given $H + K = \operatorname{Span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_q\}$.