Book Homework #7 Answers

Math 217 W11

4.2.26.

a) True. (Theorem 2)

b) True. (Theorem 3)

c) False. (Col A is the set of all $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.)

d) True. (Bottom of page 232.)

e) False. (Top of page 233.)

f) True. (e.g., example 9.)

4.2.30. Evidently $\operatorname{im} T \subseteq W$. Let $\boldsymbol{y}_1, \boldsymbol{y}_2$ be any elements of $\operatorname{im} T$, so that there exist $\boldsymbol{x}_1, \boldsymbol{x}_2 \in V$ such that $\boldsymbol{y}_1 = T(\boldsymbol{x}_1)$, $\boldsymbol{y}_2 = T(\boldsymbol{x}_2)$, and let c_1, c_2 be any scalars. Then

$$c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 = c_1T(\boldsymbol{x}_1) + c_2T(\boldsymbol{x}_2) = T(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2) \in \operatorname{im} T,$$

so that $\operatorname{im} T$ is closed under vector space operations. Also, $\mathbf{0}_W = T(\mathbf{0}_V) \in \operatorname{im} T$. Thus $\operatorname{im} T$ is a subspace of W.

4.2.31.

a) Consider two arbitrary polynomials in P_2: $\boldsymbol{p}(t) = at^2 + bt + c$, $\boldsymbol{q}(t) = dt^2 + et + f$. Then, for any scalars $r, s \in \mathbb{R}$, we have $(r\boldsymbol{p} + s\boldsymbol{q})(t) = (ra + sd)t^2 = (rb + se)t + (rc + sf)$. Now we compute as follows, verifying that T is linear.

$$T(\boldsymbol{p}) = \begin{pmatrix} c \\ a+b+c \end{pmatrix} \qquad T(\boldsymbol{q}) = \begin{pmatrix} f \\ d+e+f \end{pmatrix}$$

$$T(r\boldsymbol{p} + s\boldsymbol{q}) = \begin{pmatrix} rc + df \\ ra + rb + rc + sd + se + sf \end{pmatrix} = rT(\boldsymbol{p}) + sT(\boldsymbol{q})$$

b) A polynomial $\boldsymbol{p}$ is in $\ker T$ if 0 and 1 are both roots of the polynomial, i.e., if $t(t-1)$ divides $\boldsymbol{p}(t)$. Since we are interest only in polynomials of degree at most 2, $\ker T$ consists of exactly the constant multiples of $t(t-1)$. The polynomial $\boldsymbol{p}(t) = t^2 - t$ spans $\ker T$.

For any $a, b \in \mathbb{R}$, the polynomial $\boldsymbol{p}_{a,b}(t) = (b-a)t + a$ belongs to P_2 and has the property that $T(\boldsymbol{p}_{a,b}) = \begin{pmatrix} a \\ b \end{pmatrix}$, so $\operatorname{im} T$ is all of $\mathbb{R}^2$.

4.2.32. A polynomial $\boldsymbol{p}$ is in $\ker T$ iff $\boldsymbol{p}(0) = 0$, so the kernel consists of those polynomials with zero constant term.

$$\ker T = \{\boldsymbol{p}(t) = at^2 + bt \mid a, b \in \mathbb{R}\}$$

The polynomials $\boldsymbol{p}_1(t) = t$ and $\boldsymbol{p}_2(t) = t^2$ span $\ker T$. On the other hand, since $\boldsymbol{p}(0)$ could be any real number, $\operatorname{im} T$ consists of all vectors in $\mathbb{R}^2$ with both components the same.

$$\operatorname{im} T = \left\{ \begin{pmatrix} r \\ r \end{pmatrix} \middle| r \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

4.2.36. From the definition of U we have $U \subseteq V$. Since $T(\mathbf{0}_V) = \mathbf{0}_W$ and since Z, being a subspace of W, contains $\mathbf{0}_W$, we see $\mathbf{0}_V \in U$. Now let $\boldsymbol{x}, \boldsymbol{y}$ be arbitrary elements of U and let c, d be arbitrary scalars. Then $T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y})$. Since $\boldsymbol{x}, \boldsymbol{y} \in U$, we know $T(\boldsymbol{x}), T(\boldsymbol{y}) \in Z$. Since Z is a subspace, Z is closed under linear combinations, and $T(c\boldsymbol{x} + d\boldsymbol{y}) \in Z$. Thus $c\boldsymbol{x} + d\boldsymbol{y}$ is in U, as desired. This proves that U is a subspace of V.

4.3.10. The reduced echelon form of the matrix is $\begin{pmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$, so that a basis for the nullspace is $\left\{ \begin{pmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$.

4.3.16. Label the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$. Then row-reduce the matrix $(\boldsymbol{a} \; \boldsymbol{b} \; \boldsymbol{c} \; \boldsymbol{d} \; \boldsymbol{e})$ to obtain the matrix $\begin{pmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Thus $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a basis of the space. (In fact, any three of these vectors would form a basis, except $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}$.)

4.3.22.

a) False. The set must also span H.

b) True. (Theorem 5)

c) True. (Paragraphs preceding Example 10.)

d) False.

e) False. (The columns of A corresponding to the pivot columns of B form a basis of Col A. Columns of B are not, in general, even in Col A.)

4.3.30. By Theorem 1.8, the set cannot be linearly independent.

4.3.32. Suppose that T is one-to-one, and $\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_p)\}$ is a dependent set. Then there exist scalars c_i, not all 0, such that $\sum_{i=1}^{p} c_i T(\boldsymbol{v}_i) = \mathbf{0}$. Then we have

$$T\left(\sum_{i=1}^{p} c_i \boldsymbol{v}_i \right) = \sum_{i=1}^{p} c_i T(\boldsymbol{v}_i) = \mathbf{0} = T(\mathbf{0}),$$

which, since T is one-to-one, implies $\sum_{i=1}^{p} c_i \boldsymbol{v}_i = \mathbf{0}$. Thus the $\boldsymbol{v}_i$ are linearly dependent.

4.3.34. $\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_3 = 0$. The set $\{\boldsymbol{p}_1, \boldsymbol{p}_2\}$ is a basis for Span $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}$.

4.4.10. $\begin{pmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 2 & -5 & 7 \end{pmatrix}$

4.4.14. Relative to the standard basis, the given basis vectors have coordinates $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, and our vector of interest has coordinates $\begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix}$. Thus, in the given basis, $\boldsymbol{p}$ has the following coordinates.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -2 \end{pmatrix}$$

4.4.21. We know that $A=\begin{pmatrix} 1 & -2 \\ -4 & 9 \end{pmatrix}$ implements the inverse map to what we want, so the coordinate map is represented by $A^{-1}=\begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix}$.

4.4.28. Linearly dependent because the coordinate vectors $\begin{pmatrix} 1 \\ 0 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}$ are linearly dependent.

4.4.32.

a) Use the coordinate system given by the standard basis of P_2, so that our polynomials have coordinates $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$. Our polynomials form a basis of P_2 because these coordinates form a basis of $\mathbb{R}^3$ (one way to verify this is to compute that $\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 2 & 3 & -4 \end{pmatrix} \neq 0$).

b) $\boldsymbol{q} = -3\boldsymbol{p}_1 + \boldsymbol{p}_2 + 2\boldsymbol{p}_3$, so $\boldsymbol{q}(t) = 1 + 3t - 8t^2$.