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SCORES

Name, Section

MIDTERM 1

Math 217-W11, Linear Algebra

Directions. You have 110 minutes to complete the following 8 problems. A complete answer will always include some kind of work or justification, even for the problems which are not explicitly "formal proofs". You are not permitted to use any notecards, calculators, abaci, electronic devices of any sort, boomerangs, nor kaleidoscopes. There are a total of 140 points possible on the exam, representing 17.5% of your course grade.

1. **[20 pts]** Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $x = -y$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 45° counterclockwise. Let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection to the y-axis.

   (a) Calculate the standard matrix of the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $R(\mathbf{x}) = T(S(\mathbf{x}))$.

   The standard matrices for T and S are

   $$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

   Thus R has standard matrix

   $$AB = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

   (b) Calculate the standard matrices of T^{10}, S^{10}, and U^{10}.

   We have $T^{10} = T^2$ since T^8 is the identity transformation. Thus T^{10} has standard matrix

   $$T^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

   We have that S^2 is the identity transfromation, and so S^{10} is also the identity transformation, and thus the standard matrix of S^{10} is I.
   We have that $U^2 = U$. So the standard matrix of U^{10} is the standard matrix for U, which is

   $$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   (c) Calculate the determinant of the standard matrix of the linear transformation $W : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $W(\mathbf{x}) = S(U(S(T(\mathbf{x}))))$.

   Since U is not invertible, $\det(C) = 0$. Using $\det(XY) = \det(X)\det(Y)$, we deduce that the standard matrix of W has determinant 0.

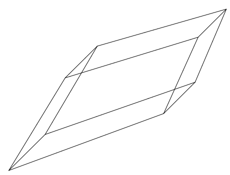
2. **[15 pts]**

(a) Find a matrix transformation $\mathbb{R}^3 \to \mathbb{R}^3$ so that the image of the cube with vertices

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$$

is the parallelepiped P with vertices at

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\4\end{bmatrix}, \begin{bmatrix}4\\3\\1\end{bmatrix}, \begin{bmatrix}5\\5\\4\end{bmatrix}, \begin{bmatrix}5\\3\\2\end{bmatrix}, \begin{bmatrix}6\\5\\5\end{bmatrix},$$



which is shown here.

Let A be the required matrix. Then A is a 3×3 matrix, and the columns of A are given by the image of the unit vectors $\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$. Thus

$$A = \begin{bmatrix}4 & 1 & 1\\3 & 0 & 2\\1 & 1 & 3\end{bmatrix}.$$

One can check that A does send every vertex of the cube correctly.

(b) Compute the volume of P using linear algebra.

The volume of P is given by the determinant of A (Theorem 9 in Chapter 3), which is -12.

3. **[20 pts]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation whose standard matrix is the following matrix A.

$$A = \begin{bmatrix} 1 & -1 & 1 & -2 & 0 \\ 2 & 1 & 5 & 2 & 1 \\ 3 & 2 & 8 & 4 & 1 \end{bmatrix}$$

(a) What is n? What is m?

$n = 5$ and $m = 3$.

(b) Compute the reduced echelon form of A.

$$\begin{bmatrix} 1 & -1 & 1 & -2 & 0 \\ 2 & 1 & 5 & 2 & 1 \\ 3 & 2 & 8 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & -2 & 0 \\ 0 & 3 & 3 & 6 & 1 \\ 0 & 5 & 5 & 10 & 1 \end{bmatrix} \longrightarrow$$
$$\begin{bmatrix} 1 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 2 & 1/3 \\ 0 & 5 & 5 & 10 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 \\ 0 & 1 & 1 & 2 & 1/3 \\ 0 & 0 & 0 & 0 & -2/3 \end{bmatrix} \longrightarrow$$
$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 \\ 0 & 1 & 1 & 2 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Is T one-to-one, onto, both, or neither?

T contains a pivot in every row, but not a pivot in every column. Thus P is onto, but not one-to-one.

(d) Let $\mathbf{a}_1$ be the first column of A. Is $T(\mathbf{x}) = \mathbf{a}_1$ consistent? If so, is the solution to $T(\mathbf{x}) = \mathbf{a}_1$ unique? If not, how many free variables are there?

Yes, $T(\mathbf{x}) = \mathbf{a}_1$ is consistent, since $T(\mathbf{e}_1) = \mathbf{a}_1$. The solution is not unique and there are three free variables.

(e) Give a vector $\mathbf{b} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{b}$ is inconsistent, or explain why there is no such vector.

T is onto. Thus no such vector exists.

4. **[30 pts]** Decide whether each of these statements is true or false. Justify your answers with short explanations or counterexamples.

(a) If A and B are square matrices of the same size, then $(A+B)^2 = A^2 + 2AB + B^2$.

False. For example, take

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(b) Suppose A and B are square and $A^2 = B^2$. Then $A = B$ or $A = -B$.

False. For example, take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) If A, B are $n \times n$ matrices such that $A^T = A$ and $B^T = B$ and if $C = ABABABA$, then $C^T = C$.

True. Using $(XY)^T = Y^T X^T$, we calculate

$$C^T = A^T B^T A^T B^T A^T B^T A^T = ABABABA = C.$$

(d) If A and B are matrices such that AB and BA are both defined, then AB and BA are the same shape.

False. For example, take

$$A = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(e) If $T : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation such that the equation $T(\mathbf{x}) = \mathbf{0}$ only has the trivial solution $\mathbf{x} = \mathbf{0}$, then T is onto.

True. The stated condition implies (by Theorem 11 in Chapter 1) that T is one-to-one. By the Inverse Matrix Theorem, T is onto.

(f) If A is a square $n \times n$ matrix such that $A^T = -A$, then $\det A = 0$.

False. The matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has non-zero determinant and satisfies $A^T = -A$.

5. **[15 pts]** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

(a) Write down what it means for T to be one-to-one.

T is one-to-one if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$.

(b) Write down what it means for $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ to be a linearly independent.

$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linearly independent if the only solution to

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = 0$$

is the trivial solution $c_1 = c_2 = c_3 = 0$.

(c) Assume that T is one-to-one. Prove that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set if and only if $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ is a linearly independent set.

Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set. If

$$c_1T(\mathbf{u}) + c_2T(\mathbf{v}) + c_3T(\mathbf{w}) = 0,$$

then by linearity of T we have

$$T(c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}) = 0$$

and using the fact that T is one-to-one, we have

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = 0$$

and so $c_1 = c_2 = c_3$ using the assumption. Thus $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ is a linearly independent set. Conversely, suppose $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ is a linearly independent set. If

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = 0,$$

then applying T to both sides and using linearity of T,

$$c_1T(\mathbf{u}) + c_2T(\mathbf{v}) + c_3T(\mathbf{w}) = 0,$$

and so $c_1 = c_2 = c_3$ using the assumption. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set.

6. **[16 pts]** Give all possible reduced echelon forms of a matrix A which satisfies the given property. (The two parts are independent.) As in the textbook, use a $\star$ to represent an arbitrary real number.

(a) A is a 5×5 matrix, and $\det A = 14$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) The matrix equation $A\mathbf{x} = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ has the unique solution $\mathbf{x} = \left[\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right]$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

7. **[14 pts]** Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^n$ are linear transformations such that $S(T(\mathbf{x})) = \mathbf{x}$ for every vector $\mathbf{x} \in \mathbb{R}^n$. Prove that T is one-to-one and that S is onto.

   We first show that T is one-to-one. Suppose that $T(\mathbf{x}) = T(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then applying S to both sides, we get $S(T(\mathbf{x})) = S(T(\mathbf{y}))$ and so $\mathbf{x} = \mathbf{y}$ by the property given in the problem. Thus T is one-to-one.

   Now we show that S is onto. Let $\mathbf{b} \in \mathbb{R}^n$. Then $\mathbf{b} = S(T(\mathbf{b}))$ so is the image of the vector $T(\mathbf{b})$ under S. Thus S is onto.

8. Warning: **This is the hardest problem on the exam. You probably want to do all you can in the rest of the exam before proceeding.**

[10 pts] Let A be a 2×2 matrix. Prove that there is a non-zero polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

so that

$$p(A) = A^n + a_{n-1} \cdot A^{n-1} + \cdots + a_1 \cdot A + a_0 \cdot I = 0.$$

For example, if $A = I$ is the identity matrix, then possible polynomials include $p(t) = t - 1$, $p(t) = t^2 - 2t + 1$, or $p(t) = t^8 - 5t^6 + 4t^5 - 3t^2 + 2t + 1$. (Of course, different matrices A can have different polynomials $p(t)$.)

Proof 1. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}.$$

By some ingenuity, we see that

$$A^2 - (a + d)A + (ad - bc)I = 0$$

so the polynomial $p(t) = t^2 - (a + d)t + (ad - bc)$ works.

Proof 2.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \qquad A^3 = \begin{bmatrix} i & j \\ k & l \end{bmatrix} \qquad A^2 = \begin{bmatrix} m & n \\ o & p \end{bmatrix}$$

Define vectors

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} i \\ j \\ k \\ l \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} m \\ n \\ o \\ p \end{bmatrix}$$

in $\mathbb{R}^4$. By Theorem 8 in Chapter 1 and the fact that $5 > 4$, the vectors $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. Thus one can find a_0, a_1, a_2, a_3, a_4, not all zero, so that $a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = 0$. Then

$$a_4A^4 + a_3A^3 + a_2A^2 + a_1A + a_0I = 0$$

Thus the polynomial

$$p(t) = t^4 + \frac{a_3}{a_4}t^3 + \frac{a_2}{a_4}t^2 + \frac{a_1}{a_4}t^1 + \frac{a_0}{a_4}$$

works, as long as $a_4 = 0$. If $a_4 = 0$, then we can use a similar polynomial of lower degree.