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SCORES

Name, Section

MIDTERM 2

Math 217-W11, Linear Algebra

Directions. You have 110 minutes to complete the following 8 problems. A complete answer will always include some kind of work or justification, even for the problems which are not explicitly "formal proofs". You are not permitted to use any notecards, calculators, abaci, electronic devices of any sort, koalas, nor kangaroos. There are a total of 140 points possible on the exam, representing 17.5% of your course grade.

1. (15 pt) Let T be the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ with matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

(a) Compute the dimension of the kernel of T and a basis for the kernel.

The kernel of T is the nullspace of A, so we row reduce A to find

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence a basis for the kernel consists of the one vector

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which means that the kernel of T is one-dimensional.

(b) Compute the rank of A and a basis for the image of T.

We already row reduced A and we found two pivots, which means that the rank of A is 2.

The image of T is the column space of A and a basis for that space consists of the first two columns of A:

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. (15 pt) Let

$$A = \begin{bmatrix} -10 & -18 \\ 9 & 17 \end{bmatrix}.$$

(a) Compute the eigenvalues of A.

The characteristic equation of A is

$$\det(A - \lambda I) = \begin{vmatrix} -10-\lambda & -18 \\ 9 & 17-\lambda \end{vmatrix} = \lambda^2 - 7\lambda - 8.$$

The roots of this equation are -1 and 8, which are therefore the eigenvalues of A.

(b) Compute a basis for each eigenspace of A.

For the eigenvalue -1, the eigenspace is the nullspace of

$$\begin{bmatrix} -9 & -18 \\ 9 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Therefore a basis for this eigenspace consists of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, for example.

For the eigenvalue 8, the eigenspace is the nullspace of

$$\begin{bmatrix} -18 & -18 \\ 9 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore a basis for this eigenspace consists of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, for example.

(c) Compute a matrix B such that $B^3 = A$.

We know that $A = PDP^{-1}$ where $D = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}$ and $P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$. Consider the diagonal matrix $E = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Note that $E^3 = D$ and hence $B^3 = A$ where $B = PEP^{-1}$. Thus, the matrix

$$B = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

works!

3. (15 pt) Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by

$$T(p(t)) = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix}$$

for every polynomial $p(t) = a_2t^2 + a_1t + a_0$ in $\mathbb{P}_2$.

(a) Show that T is a linear transformation.

We must show that $T((p+q)(t)) = T(p(t)) + T(q(t))$ and $T((cp)(t)) = cT(p(t))$ hold for all polynomials $p(t)$ and $q(t)$ in $\mathbb{P}_2$ and every scalar $c \in \mathbb{R}$.

Both equalities are immediate since $(p+q)(t)$ is defined to equal $p(t)+q(t)$ and $(cp)(t)$ is defined to equal $c(p(t))$ for every value of t.

(b) Find a matrix M such that $T(p(t)) = M[p(t)]_{\mathcal{B}}$ where $\mathcal{B} = \{1, t, t^2\}$.

Write $p(t) = a_2t^2 + a_1t + a_0$ so that $[p(t)]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. Then

$$\begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix} = \begin{bmatrix} a_2 + a_1 + a_0 \\ a_2 4 + a_1 2 + a_0 \\ a_2 9 + a_1 3 + a_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix},$$

which shows that $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ works.

(c) Show that for all $c_1, c_2, c_3 \in \mathbb{R}$ there is a polynomial $p(t) \in \mathbb{P}_2$ such that $p(1) = c_1$, $p(2) = c_2$, and $p(3) = c_3$.

We are asked to solve the equation $T(p(t)) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ for the polynomial $p(t) = a_2t^2 + a_1t + a_0$. By part (b), this equation is equivalent to the matrix equation

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

This equation always has a solution since the matrix M is invertible.

4. (15 pt) Let V be a vector space (not necessarily finite-dimensional), and let $\mathbf{0}$ be the zero vector of V.

(a) Suppose that $\mathbf{w} \in V$ is a vector such that $\mathbf{w} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$. Prove that $\mathbf{w} = \mathbf{0}$. That is, prove that the zero vector is the *unique* vector in V that satisfies its defining property.

By Axiom 5 (page 217) we know that there is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. Adding $-\mathbf{v}$ to both sides of $\mathbf{w} + \mathbf{v} = \mathbf{v}$, we obtain

$$(\mathbf{w} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{v} + (-\mathbf{v}).$$

The right hand side equals $\mathbf{0}$ by the defining property of $-\mathbf{v}$. For the left hand side, we have

$$(\mathbf{w} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{w} + (\mathbf{v} + (-\mathbf{v}))$$

by Axiom 3. Since $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, we then see that

$$\mathbf{w} + (\mathbf{v} + (-\mathbf{v})) = \mathbf{w} + \mathbf{0} = \mathbf{w}$$

by Axiom 4. Putting things together, we obtain that $\mathbf{w} = \mathbf{0}$.

(b) Suppose that $\mathbf{v} \in V$, and that $\mathbf{w} \in V$ is a vector such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. Prove that $\mathbf{w} = -\mathbf{v}$. That is, prove that $-\mathbf{v}$ is the *unique* vector in V that satisfies its defining property.

We know by Axiom 5 that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. Adding $-\mathbf{v}$ to both sides of $\mathbf{v} + \mathbf{w} = \mathbf{0}$, we obtain

$$(-\mathbf{v}) + (\mathbf{v} + \mathbf{w}) = (-\mathbf{v}) + \mathbf{0}.$$

The right hand side equals $-\mathbf{v}$ by Axiom 4. For the left hand side, we have

$$(-\mathbf{v}) + (\mathbf{v} + \mathbf{w}) = ((-\mathbf{v}) + \mathbf{v}) + \mathbf{w} = (\mathbf{v} + (-\mathbf{v})) + \mathbf{w}$$

by Axioms 3 and 2. Since $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ by Axiom 5, we further obtain

$$(\mathbf{v} + (-\mathbf{v})) + \mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{w} + \mathbf{0} = \mathbf{w}$$

by Axioms 2 and 4. Putting things together, we obtain $\mathbf{w} = -\mathbf{v}$.

5. (30 pt) Decide whether each of these statements is true or false. Justify your answers with short explanations or counterexamples.

(a) If A and B both have the characteristic polynomial $-\lambda^3 + 7\lambda^2 - 10\lambda$, then A and B are similar matrices.

True. The characteristic polynomial has three distinct roots: 0, 2, 5. Therefore A and B are both similar to the diagonal matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, which means that they are also similar to eachother.

(b) If V and W are vector spaces of dimension n and $T : V \to W$ is a one-to-one linear transformation, then T is onto.

True. Because T is one-to-one, the dimension of the image of T must be n. Since W has dimension n, the image of T must equal W.

(c) If A is an $n \times n$ matrix and $\mathbf{x}, \mathbf{y}$ are two eigenvectors of A, then $\mathbf{x} + \mathbf{y}$ is also an eigenvector of A.

False. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both eigenvectors of A, but $\mathbf{x} + \mathbf{y}$ is not an eigenvector of A since

$$A(\mathbf{x} + \mathbf{y}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is not a multiple of $\mathbf{x} + \mathbf{y}$.

(d) If P is a stochastic matrix, then 1 is an eigenvalue of P^T. (Recall that a square matrix is stochastic if every column of it is a probability vector.)

True. The easiest way to prove this is to show that the vector $\mathbf{j} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector of P^T with eigenvalue 1. Indeed, the coordinates of $P^T\mathbf{j}$ are simply the sum of the entries of each row of P^T. Since the rows of P^T are probability vectors, the sum of these entries is always 1. Therefore $P^T\mathbf{j} = \mathbf{j}$, which shows that $\mathbf{j}$ is an eigenvector with eigenvalue 1.

(e) If $\mathbf{x}$ is an eigenvector of a matrix A, then $\mathbf{x}$ is also an eigenvector of A^3.
True. Suppose that $A\mathbf{x} = \lambda\mathbf{x}$. Then

$$A^3\mathbf{x} = A^2(A\mathbf{x}) = A^2(\lambda\mathbf{x}) = \lambda A^2\mathbf{x} = \lambda A(A\mathbf{x}) = \lambda A(\lambda\mathbf{x}) = \lambda^2 A\mathbf{x} = \lambda^3\mathbf{x},$$

which shows that $\mathbf{x}$ is an eigenvector of A^3 with eigenvalue λ^3.

(f) There is a 2×4 matrix A with rank 1 and

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

True. An example of such a matrix is

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, a basis for the eigenspace of this matrix consists of the three vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

which is also a basis of

$$\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

since the third spanning vector is redundant.

6. (20 pt)

(a) Give an example of a real $n \times n$ matrix A with eigenvalues $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$.
Any real 2×2 matrix with characteristic equation

$$\left(\frac{-1+i\sqrt{3}}{2} - \lambda\right)\left(\frac{-1-i\sqrt{3}}{2} - \lambda\right) = \lambda^2 + \lambda + 1$$

will do. One such matrix is $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.

(b) Give a complete list of all diagonalizable matrices with characteristic polynomial $(\lambda - 2)^4$. You must prove your list is complete.
One such matrix is the diagonal matrix

$$2I_{4\times 4} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Indeed, any diagonalizable matrix with characteristic polynomial $(\lambda - 2)^4$ will be similar to this one. However, if P is any 4×4 invertible matrix then $P(2I_{4\times 4})P^{-1} = 2PI_{4\times 4}P^{-1} = 2PP^{-1} = 2I_{4\times 4}$. Therefore $2I_{4\times 4}$ is the only diagonalizable matrix with characteristic polynomial $(\lambda - 2)^4$.

(c) Find the algebraic multiplicities, and the dimension of the eigenspaces, of all the eigenvalues of the linear transformation $D : \mathbb{P}_2 \to \mathbb{P}_2$ which sends each polynomial $p(t) = a_2t^2 + a_1t + a_0$ in $\mathbb{P}_2$ to its derivative $p'(t) = 2a_2t + a_1$.

The standard basis for $\mathbb{P}_2$ is $\mathcal{B} = \{1, t, t^2\}$. The $\mathcal{B}$-matrix of D is

$$[D]_{\mathcal{B}} = \begin{bmatrix}[D(1)]_{\mathcal{B}} & [D(t)]_{\mathcal{B}} & [D(t^2)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix is upper triangular, we can read off the eigenvalues of D: 0 is the only eigenvalue, with algebraic multiplicity 3.

Since the nullspace of $[D]_{\mathcal{B}}$ has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we see that the only eigenvectors of D are the nonzero constant polynomials. Therefore, the eigenspace of D with eigenvalue 0 is one-dimensional.

7. (15 pt) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for $\mathbb{R}^3$. Suppose that A and B are 3×3 matrices such that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are eigenvectors of both A and B. Show that $AB = BA$.

Let $P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$. Then

$$A = PDP^{-1}, \qquad B = PEP^{-1}$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad E = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}.$$

Then

$$AB = (PDP^{-1})(PEP^{-1}) = PDEP^{-1} = PEDP^{-1} = (PDP^{-1})(PEP^{-1}) = BA,$$

since

$$DE = \begin{bmatrix} \lambda_1\mu_1 & 0 & 0 \\ 0 & \lambda_2\mu_2 & 0 \\ 0 & 0 & \lambda_3\mu_3 \end{bmatrix} = ED.$$

8. (15 pt) Let A be an $n \times n$ matrix which is not invertible, and define K_r to be the nullspace of A^r. That is, $K_1 = \text{Nul}(A)$, $K_2 = \text{Nul}(A^2)$, etc.

(a) Prove that, for each $j = 1, 2, 3, \ldots$, K_j is a subspace of K_{j+1}, so that

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4 \subseteq \cdots.$$

We need to show that $K_j \subseteq K_{j+1}$. So suppose $\mathbf{x} \in K_j$, that is $A^j\mathbf{x} = \mathbf{0}$. Then

$$A^{j+1}\mathbf{x} = A(A^j\mathbf{x}) = A\mathbf{0} = \mathbf{0},$$

which shows that $\mathbf{x} \in K_{j+1}$.

(b) Prove that, if for some r we have $K_r = K_{r+1}$, then $K_r = K_n$ for all $n \geq r$.

Suppose that $K_r = K_{r+1}$. We prove by induction on $n \geq 1$ that $K_r = K_{r+n}$.

The base case $n = 1$ is given to us!

For the induction step, suppose we have shown that $K_r = K_{r+n}$. To show that $K_r = K_{r+n+1}$ we can show instead that $K_{r+n} = K_{r+n+1}$. We already know that $K_{r+n} \subseteq K_{r+n+1}$ so it suffices to show that $K_{r+n+1} \subseteq K_{r+n}$.

Suppose that $\mathbf{x} \in K_{r+n+1}$, that is $A^{r+n+1}\mathbf{x} = \mathbf{0}$. Then $A^{r+1}(A^n\mathbf{x}) = \mathbf{0}$, which means that $A^n\mathbf{x} \in K_{r+1}$. Since $K_{r+1} = K_r$, we conclude that $A^n\mathbf{x} \in K_r$ and hence $A^r(A^n\mathbf{x}) = \mathbf{0}$. Since $A^rA^n = A^{n+r}$, we conclude that $\mathbf{x} \in K_{n+r}$.