Practice Problems for Final Exam

Math 217, Linear Algebra

(1) Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 2 & 2 & 0 \\ -1 & 2 & 4 & 0 \end{bmatrix}$.

(a) Compute the reduced row echelon form of A.
(b) Find a basis for $\ker(A)$ and for $\operatorname{Im}(A)$.

(2) Let W be the plane $x - z = 0$ in $\mathbb{R}^3$.

(a) Find an orthonormal basis for W.
(b) Find the standard matrix R for reflections of vectors over W.
(c) Diagonalize R (Hint: try to understand R from geometry).
(d) Is R invertible?

(3) Suppose Q is a 12×12 orthogonal matrix and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ -1 \end{bmatrix}$ in $\mathbb{R}^{12}$.

Find $||Q\mathbf{x}||$ and $\langle Q\mathbf{x}, Q\mathbf{y} \rangle$.

(4) If $\{\mathbf{u}, \mathbf{v}\}$ is an orthogonal set in $\mathbb{R}^{14}$, Q is an orthogonal matrix in $M_{14\times14}$, and the distance from $\mathbf{u}$ to $-\mathbf{v}$ is 4, compute the distance from $Q\mathbf{u}$ to $Q\mathbf{v}$.

(5) In the QR factorization of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & 0 \end{bmatrix}$ (where $\sin(\theta) \neq 0$), find Q and check that $Q^T = Q^{-1}$.

(6) Let A be a 2×2 matrix. Suppose A has an eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue -2 and another eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with eigenvalue 3.

(a) Find A.
(b) Find a closed formula for A^n where n is a positive integer.

(7) Let Poly_1 be the space of all linear functions, i.e. functions of the form $f(x) = ax + b$, where a, b are real numbers. Consider the linear transformation $T : \mathsf{Poly}_1 \to \mathsf{Poly}_1$ that takes f to $-3f' + f$.

(a) Find the matrix of T relative to the standard basis $\{1, x\}$.
(b) Find the matrix of T relative to the basis $\{1 - x, 1 + x\}$.
(c) What are the determinants of those matrices? Is the transformation invertible?

(8) (For this problem, imagine that there's generous partial credit for doing the parts for a single example $\mathbf{u} \in \mathbb{R}^2$.)

Let $\mathbf{u} \in \mathbb{R}^n$ be a column vector with unit length and define $H = I - \mathbf{u}\mathbf{u}^T$. (*Not* $\mathbf{u}^T\mathbf{u}$!)

(a) Prove that H is symmetric and $H^2 = H$.
(b) Prove that $\mathbf{u}$ is an eigenvector and find its corresponding eigenvalue.
(c) Prove that whenever $\mathbf{v} \perp \mathbf{u}$, then $\mathbf{v}$ is also an eigenvector.

(d) Prove that H is diagonalizable.

(9) Recall that the *trace* of a square matrix is the sum of the entries on the main diagonal. Prove that similar square matrices have the same trace.

(10) **Give an example if possible. If not, say why not.**
(a) A 5×4 matrix with with no zero entries with rank and nullity both 2.
(b) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors.
(c) A 2×3 matrix Q such that $Q^T Q = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$.
(d) A linear transformation $T : \mathsf{Poly}_2 \to \mathbb{R}^3$ such that $T(x) = 0$ and T is onto.
(e) A linear transformation $T : \mathbb{R}^3 \to \mathsf{Poly}_2$ such that $T(e_1) = x^2 + x + 1$ and T is onto.

(11) **Short proofy problems.**
(a) Consider $M_{2\times 2}$ with the standard basis $\mathcal{Q}$, and $T : M_{2\times 2} \to M_{2\times 2}$ be given by swapping rows. Prove that T is linear and find its $\mathcal{Q}$-matrix.
(b) Let λ be an eigenvalue of a square matrix A. Show that $\lambda^3 + 5\lambda + 1$ is an eigenvalue of the matrix $A^3 + 5A + I$.
(c) Find all possible determinants of all orthogonal $n \times n$ matrices for all n.
(d) Let V be a vector space and $S, T : V \to V$ be linear transformations such that $\mathbf{x}$ is an eigenvector for both S and T. Prove that $\mathbf{x}$ is also an eigenvector for $S \circ T$.
(e) Let $H \subset K \subset \mathbb{R}^n$ be subspaces. Prove that $K^\perp \subset H^\perp$.
(f) Show that if $\mathbf{w}, \mathbf{v} \in V$ are vectors such that $\langle \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{x} \rangle$ for all $\mathbf{x} \in V$, then $\mathbf{w} = \mathbf{v}$.
(g) Suppose $A \in M_{4\times 4}$ has complex eigenvalues $\lambda = 3 + i$ and $\lambda = 4 + 2i$. Show that A has no eigenvectors in $\mathbb{R}^4$.

(12) **Grammar check.** Meaningful/nonsense? If meaningful, true/false? If nonsense or false, replace with a similar true statement.
(a) The square of a vector is always nonnegative.
(b) If W is a two-dimensional subspace of $\mathbb{R}^3$, then $W^\perp$ has one element.
(c) If an elementary matrix is applied to every vector in a basis $\mathcal{B}$ for $\mathbb{R}^n$, then the output vectors also form a basis.
(d) If $A, B \in M_{2\times 2}$ satisfy $AB = 0$, then $BA = I$.
(e) If you apply the Gram-Schmidt method to the entries of $\left[\begin{smallmatrix} 5 & 7 \\ 0 & 4 \end{smallmatrix}\right]$, it will result in a factorization into $\left[\begin{smallmatrix} 5 & 0 \\ 0 & 4 \end{smallmatrix}\right] \begin{bmatrix} 1 & -7/5 \\ 0 & 1 \end{bmatrix}$.

(13) (Input-Output Economics Problem)
Suppose you want to study the paper sector and the electronics sector of your economy. Suppose the inter-industry consumption matrix and the final demand vector are $C = \begin{bmatrix} .1 & .6 \\ .5 & .2 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 10 \\ 22 \end{bmatrix}$. Solve the production equation. Give an interpretation of all the numbers in C, $\mathbf{d}$, and $\mathbf{x}$. Which industry is more efficient?

(14) (Markov Chains Problem)
Suppose you consider five possible mood statuses (happy, hungry, dazed, incensed, and ticklish), and you form a 5×5 stochastic matrix P describing the transition probabilities

between these moods from one moment to the next. If $\mathbf{x}$ is an eigenvector of P corresponding to the eigenvalue $\lambda = 1$, give an interpretation of the meaning of $\mathbf{x}$ in terms of your moods.

(15) (Least Squares Problem) Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}.$$