Proofs Homework Set 2

MATH 217 — WINTER 2011

Due January 19

Functions

Let X and Y be sets. A **function** $f : X \to Y$ is a map which assigns a unique element $f(x) \in Y$ to each element $x \in X$. The **domain** of f is the set X; the **codomain** of f is the set Y.
Let $A \subseteq X$. The **image** of A under f is the set

$$f(A) = \{f(x) \mid x \in A\}.$$

Let $B \subseteq Y$. The **preimage** of B under f is the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

PROBLEM 2.1. Decide whether or not each of the following is a function. Justify your answers.

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + x - 5$.

Answer. This is a function. Every $x \in \mathbb{R}$ is assigned a unique value, namely $x^3 + x - 5$.

(b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 & \text{if } x \geq 2, \\ 3x - 1 & \text{if } x \leq 2. \end{cases}$

Answer. This is not a function. The real number $x = 2$ is assigned two values: $2^2 = 4$ and $3 \cdot 2 - 1 = 5$.

(c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 & \text{if } x \geq 2, \\ 3x - 2 & \text{if } x \leq 2. \end{cases}$

Answer. This is a function. Every $x \in \mathbb{R}$ is assigned a unique value, including $x = 2$ which is assigned the value $2^2 = 4 = 3 \cdot 2 - 2$.

PROBLEM 2.2. Decide whether the following statements are true or false. If true, prove it. If false, provide a counterexample which shows that the statement is false; *i.e.* give an explicit, concrete example of a function f for which the equality fails—don't forget to provide the domain and codomain in your example!

(a) $f^{-1}(f(A)) = A$.

Answer. False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $A = [0, \infty)$. By the definition of image, $f(A) = [0, \infty) = A$. However, $f^{-1}(f(A)) = f^{-1}(A) = \mathbb{R}$, which is not equal to A.

(b) $f(f^{-1}(B)) = B$.

Answer. False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $B = \mathbb{R}$. By the definition of preimage, $f^{-1}(B) = \mathbb{R}$. However, $f(f^{-1}(B)) = f(\mathbb{R}) = [0, \infty)$, which is not equal to B.

(c) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Answer. True. To prove this, the strategy is to show $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$ and $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.
For the first containment: For any element $y \in f(A_1 \cup A_2)$, there exists an element $x \in A_1 \cup A_2$ such that $f(x) = y$. By the definition of union, $x \in A_1$ or $x \in A_2$. This implies that $y \in f(A_1)$ or $y \in f(A_2)$. Therefore, $y \in f(A_1) \cup f(A_2)$.
For the second containment: Let $y \in f(A_1) \cup f(A_2)$. Then $y \in f(A_1)$ or $y \in f(A_2)$ so there is $x_1 \in A_1$ such that $f(x_1) = y$ or there is $x_2 \in A_2$ such that $f(x_2) = y$. In either case, we have an element x in $A_1 \cup A_2$ such that $f(x) = y$. Therefore $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.
It follows that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(d) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

Answer. False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $A_1 = [0, \infty)$ and $A_2 = (-\infty, 0]$. Then $A_1 \cap A_2 = \{0\}$ so $f(A_1 \cap A_2) = \{0\}$. However, $f(A_1) = f(A_2) = [0, \infty)$, so $f(A_1) \cap f(A_2) = [0, \infty)$. Note that this happened because two different elements in the domain are mapped to the same element in the range.

Induction

The set of positive integers $\{1, 2, 3, \dots\}$ is called the **natural numbers** and is denoted by $\mathbb{N}$. Mathematical statements which depend on a natural number n can sometimes be proved using the method of induction. We state the principle of induction and provide an example in which we employ this principle to prove a statement below:

PRINCIPLE OF MATHEMATICAL INDUCTION. *Consider a statement $P(n)$ which depends on a natural number $n \in \mathbb{N}$. If*

(i) *$P(k)$ is true, and*

(ii) *For any given $n \in \mathbb{N}$ such that $n \geq k$, we have $P(n)$ true $\Longrightarrow P(n+1)$ true,*

then the statement $P(n)$ is true for all $n \geq k$.

EXAMPLE. Prove that the sum $\underbrace{1 + 2 + \cdots n = \frac{n(n+1)}{2}}_{\text{statement } P(n)}$ for all $n \in \mathbb{N}$.

Proof. The base case is the statement for $n = 1$. When $n = 1$, we have that $\frac{1(1+1)}{2} = \frac{2}{2} = 1$, and the equality holds.

For the inductive step, assume that the equality $1 + 2 + \cdots n = \frac{n(n+1)}{2}$ holds, and consider the sum $1 + 2 + \cdots + n + (n + 1)$. By the inductive hypothesis, we have

$$\begin{aligned} 1 + 2 + \cdots + n + (n + 1) &= \frac{n(n + 1)}{2} + (n + 1) \\ &= \frac{n(n + 1)}{2} + \frac{2(n + 1)}{2} \\ &= \frac{(n + 2)(n + 1)}{2} \\ &= \frac{(n + 1)((n + 1) + 1)}{2}, \end{aligned}$$

and so the equality also holds for $n + 1$.

By the Principle of Mathematical Induction, the result holds for all $n \in \mathbb{N}$. $\square$

For problems 2.3 and 2.4, use induction to prove the given statement. Carefully explain what you are doing in your proof (*e.g.* your hypotheses in each step, the conclusion you wish to draw). Begin and end your proof by mentioning that you are using or have used induction. You will be graded as much on form as on mathematical content.

PROBLEM 2.3. Prove that the sum of the first n odd natural numbers is n^2.

Proof. Note that the n^{th} odd natural number can be denoted $2n - 1$, so we are being asked to prove that $1 + 3 + \cdots + (2n - 1) = n^2$.

The base case is the statement for $n = 1$. When $n = 1$, the statement reads as $1 = 1^2$, and this equality holds.

For the inductive step, assume that the equality $1 + 3 + \cdots + (2n - 1) = n^2$ holds, and consider the sum $1 + 3 + \cdots + (2n - 1) + (2n + 1)$. By the inductive hypothesis, we have

$$\begin{aligned} 1 + 3 + \cdots + (2n - 1) + (2n + 1) &= n^2 + (2n + 1) \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2. \end{aligned}$$

By the Principle of Mathematical Induction, the result holds for all $n \in \mathbb{N}$. $\square$

PROBLEM 2.4. Prove the power rule for derivatives of polynomials:

$$\text{for } f(x) = x^n,\ f'(x) = nx^{n-1}.$$

You may use the product rule for derivatives (namely, $(fg)' = f'g + fg'$) in your argument.

Proof. The base case is the statement for $n = 1$. When $n = 1$, the statement reads as "*for* $f(x) = x^1, f'(x) = 1x^{1-1} = 1$," and this equality holds.

For the inductive step, assume that the statement "*for* $f(x) = x^n,\ f'(x) = nx^{n-1}$" holds, and consider the function $f(x) = x^{n+1}$. Since $x^{n+1} = x^n \cdot x$, the product rule, we have $(x^{n+1})' = (x^n \cdot x)' = (x^n)' \cdot x + x^n \cdot (x)'$. By the inductive hypothesis and the base case, we get

$$\begin{aligned}(x^{n+1})' &= (x^n \cdot x)' \\ &= (x^n)' \cdot x + x^n \cdot (x)' \\ &= nx^{n-1} \cdot x + x^n \cdot 1 \\ &= nx^n + x^n \\ &= (n+1)x^n.\end{aligned}$$

By the Principle of Mathematical Induction, the result holds for all $n \in \mathbb{N}$. □

PROBLEM 2.5. Find the flaw(s) in the following inductive "proof":

RIDICULOUS CLAIM. *All tables are the same height.*

"Proof". To prove this by induction, we let $P(n)$ be the statement "For any set of n tables, all n tables are the same height." If we prove this true for all n, it will certainly be true for $n =$ the number of tables that exist.

Now we proceed by induction on the number of tables. The base case is the case in which there is one table. Since this table is the same height as itself, the base case is true. Now assume that the statement holds for any set of n tables, and consider a set of $n + 1$ tables.

Put the tables in a line. If we remove the first table, we are left with a set of n tables. Then by the inductive hypothesis, these n tables must all be the same height. If, instead, we had removed the last table, we would again have n tables, which would now include the first one, and again by inductive hypothesis all n tables would be the same height. Therefore, all of the tables must be the same height as, for instance, the second table from the front, and consequently must be the same height as one another. The result then follows from induction. □

Answer. The flaw in the argument is that it is assumed that there is always a 'middle group' of tables, that is, tables that are neither first nor last and therefore common to both sets. This is not true when there are exactly 2 tables.