Proofs Homework Set 4

MATH 217 — WINTER 2011

Due February 2

PROBLEM 4.1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Prove that any three vectors in the image of T are linearly dependent.

Proof. Recall that the image of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ consists of all vectors $\mathbf{y}$ in $\mathbb{R}^m$ which are of the form $\mathbf{y} = T(\mathbf{x})$ for some $\mathbf{x}$ in $\mathbb{R}^n$.

If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ lie in the image of T, we can write

$$\mathbf{v}_1 = T(\mathbf{w}_1), \quad \mathbf{v}_2 = T(\mathbf{w}_2), \quad \mathbf{v}_3 = T(\mathbf{w}_3)$$

for some $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ in $\mathbb{R}^2$. Theorem 8 in §1.7 tells us that if a set contains more vectors than there are entries in each vector, then the set is linearly dependent. In our case, as $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are three vectors in $\mathbb{R}^2$, they must be linearly dependent, meaning that there exist real numbers c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}.$$

Applying T to both sides of this equality, and using linearity of T (namely that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$), we obtain

$$\begin{aligned} T(c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3) &= T(\mathbf{0}) \\ T(c_1\mathbf{w}_1) + T(c_2\mathbf{w}_2) + T(c_3\mathbf{w}_3) &= \mathbf{0} \\ c_1T(\mathbf{w}_1) + c_2T(\mathbf{w}_2) + c_3T(\mathbf{w}_3) &= \mathbf{0} \\ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 &= \mathbf{0}. \end{aligned}$$

This is a linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ which equals zero, and in which not all coefficients c_1, c_2, c_3 equal zero. We have found a linear dependence relation for $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, and so the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent. □

PROBLEM 4.2. Let A be a 3×4 matrix and $\mathbf{b}, \mathbf{c}$ be two vectors in $\mathbb{R}^3$ such that both matrix equations

$$A\mathbf{x} = \mathbf{b} \qquad \text{and} \qquad A\mathbf{y} = \mathbf{c}$$

are consistent. Prove that there exists a vector $\mathbf{d} \in \mathbb{R}^4$ such that the set of solutions $\mathbf{y}$ to the second equation $A\mathbf{y} = \mathbf{c}$ is the set of all vectors of the form $\mathbf{y} = \mathbf{x} + \mathbf{d}$, where $\mathbf{x}$ is any solution of the first equation $A\mathbf{x} = \mathbf{b}$.

Proof. Let $\mathbf{p}$ and $\mathbf{q}$ be some particular solutions of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{c}$, respectively; *i.e.*

$$A\mathbf{p} = \mathbf{b}, \quad A\mathbf{q} = \mathbf{c}.$$

We will prove that $\mathbf{y}$ is a solution of $A\mathbf{y} = \mathbf{c}$ if and only if $\mathbf{y} = \mathbf{x} + (\mathbf{q} - \mathbf{p})$ for some $\mathbf{x}$ which is a solution of $A\mathbf{x} = \mathbf{b}$. This will mean that the choice $\mathbf{d} = \mathbf{q} - \mathbf{p}$ is permissible.

Suppose first that $\mathbf{y}$ is a solution of $A\mathbf{y} = \mathbf{c}$. For $\mathbf{y} = \mathbf{x} + (\mathbf{q} - \mathbf{p})$ to be true, we need that $\mathbf{x} = \mathbf{y} - \mathbf{q} + \mathbf{p}$. We verify that this $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$. By using the properties of Theorem 5 in §1.4, we compute

$$\begin{aligned} A\mathbf{x} &= A(\mathbf{y} - \mathbf{q} + \mathbf{p}) \\ &= A\mathbf{y} - A\mathbf{q} + A\mathbf{p} = \mathbf{c} - \mathbf{c} + \mathbf{b} = \mathbf{b}, \end{aligned}$$

so that $\mathbf{x}$ is indeed a solution of $A\mathbf{x} = \mathbf{b}$.

Suppose next that $\mathbf{y} = \mathbf{x} + (\mathbf{q} - \mathbf{p})$ for some $\mathbf{x}$ which is a solution of $A\mathbf{x} = \mathbf{b}$; we need to verify that $\mathbf{y}$ is a solution of $A\mathbf{y} = \mathbf{c}$. Using the same properties as above,

$$\begin{aligned} A\mathbf{y} &= A(\mathbf{x} + \mathbf{q} - \mathbf{p}) \\ &= A\mathbf{x} + A\mathbf{q} - A\mathbf{p} = \mathbf{b} + \mathbf{c} - \mathbf{b} = \mathbf{c}, \end{aligned}$$

so that $\mathbf{y}$ is indeed a solution of $A\mathbf{y} = \mathbf{c}$. □