Proofs Homework Set 6

MATH 217 — WINTER 2011

Due February 16

PROBLEM 6.1.

(a) If A is an invertible $n \times n$ matrix, find $\det(A^{-1})$ in terms of $\det(A)$.

Proof. Since $AA^{-1} = I_n$, we see that

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1.$$

Since $\det(A) \neq 0$, we conclude that $\det(A^{-1}) = 1/\det(A)$. □

(b) If A and C are $n \times n$ matrices and C is invertible, show that $\det(A) = \det(CAC^{-1})$.

Proof. Since determinants are multiplicative, we have

$$\det(CAC^{-1}) = \det(C)\det(A)\det(C^{-1}).$$

By part (a), we know that $\det(C^{-1}) = 1/\det(C)$. Therefore,

$$\det(CAC^{-1}) = \frac{\det(C)}{\det(C)}\det(A) = \det(A),$$

as required. □

PROBLEM 6.2. We say that a square matrix A is **skew-symmetric** if $A^T = -A$.

Suppose that A is a skew-symmetric $n \times n$ matrix and that $\mathbf{x}$ is a solution of the homogeneous equation $(A + I_n)\mathbf{x} = \mathbf{0}$.

(a) Show that $A\mathbf{x} = -\mathbf{x}$.

Proof. Since $(A + I_n)\mathbf{x} = \mathbf{0}$, we have $A\mathbf{x} + \mathbf{x} = \mathbf{0}$ and hence $A\mathbf{x} = -\mathbf{x}$. □

(b) Show that $\mathbf{x}^T A = \mathbf{x}^T$.

Proof. Transposing $A\mathbf{x} = -\mathbf{x}$, we obtain $\mathbf{x}^T A^T = -\mathbf{x}^T$. Since $A^T = -A$, we conclude that $\mathbf{x}^T A = \mathbf{x}^T$. □

(c) Using parts (a) and (b), show that $\mathbf{x}^T\mathbf{x} = -\mathbf{x}^T\mathbf{x}$.

Proof. Multiplying the equation of part (a) on the left by $\mathbf{x}^T$, we obtain $\mathbf{x}^T A\mathbf{x} = -\mathbf{x}^T\mathbf{x}$.
Multiplying the equation of part (b) on the right by $\mathbf{x}$, we obtain $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T\mathbf{x}$.
Combining these two facts, we obtain that $\mathbf{x}^T\mathbf{x} = -\mathbf{x}^T\mathbf{x}$. □

(d) Using part (c), show that $\mathbf{x} = \mathbf{0}$.

Proof. First note that
$$\mathbf{x}^T\mathbf{x} = [x_1^2 + x_2^2 + \cdots + x_n^2].$$
It follows from part (c) that $\mathbf{x}^T\mathbf{x} = [0]$ (since 0 is the only number that equals its negative). Therefore
$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0,$$
which is only possible if $x_1 = x_2 = \cdots = x_n = 0$. □

(e) Conclude that $A + I_n$ is invertible.

Proof. It follows from the above that the only solution of the homogeneous equation $(A + I_n)\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$. Therefore, by the Invertible Matrix Theorem, the square matrix $A + I_n$ is invertible. □