Proofs Homework Set 7

MATH 217 — WINTER 2011

Due February 23

PROBLEM 7.1. Let V and W be vector spaces, and suppose that $T : V \to W$ is a one-to-one linear transformation. If there are vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V such that the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)$ span W, prove that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span V.

Proof. Pick an arbitrary $\mathbf{v} \in V$. Since the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)$ span W, there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k).$$

By applying the linearity of T to the right-hand side, this is equivalent to

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k).$$

Since T is one-to-one, every vector in W is an image of at most one vector in V under T. This means that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

This equality shows that every vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, which means that these vectors span V. □

PROBLEM 7.2. Let V be a vector space. Suppose that H is a nonempty subset of V such that $\mathrm{Span}\{\mathbf{x}, \mathbf{y}\} \subseteq H$ for all vectors $\mathbf{x}, \mathbf{y} \in H$. Prove that H is a subspace of V.

Proof. We need to verify three facts:

(1) The zero vector $\mathbf{0}$ belongs to H.

(2) If $\mathbf{x}, \mathbf{y}$ both belong to H then $\mathbf{x} + \mathbf{y}$ belongs to H too.

(3) If $\mathbf{x}$ belongs to H and $c \in \mathbb{R}$, then $c\mathbf{x}$ belongs to H too.

For (1), since H is nonempty there is a vector $\mathbf{x}$ that belongs to H. Then $\mathbf{0} = 0\mathbf{x}$ belongs to $\mathrm{Span}\{\mathbf{x}\}$. Since $\mathrm{Span}\{\mathbf{x}\} = \mathrm{Span}\{\mathbf{x}, \mathbf{x}\}$ is a subset of H, we conclude that $\mathbf{0}$ belongs to H.

For (2), suppose that $\mathbf{x}, \mathbf{y}$ both belong to H. Then $\mathbf{x} + \mathbf{y}$ belongs to $\mathrm{Span}\{\mathbf{x}, \mathbf{y}\}$. Since $\mathrm{Span}\{\mathbf{x}, \mathbf{y}\}$ is a subset of H, we conclude that $\mathbf{x} + \mathbf{y}$ belongs to H.

For (3), suppose that $\mathbf{x}$ belongs to H and that $c \in \mathbb{R}$. Then $c\mathbf{x}$ belongs to $\mathrm{Span}\{\mathbf{x}\}$. Since $\mathrm{Span}\{\mathbf{x}\} = \mathrm{Span}\{\mathbf{x}, \mathbf{x}\}$ is a subset of H, we conclude that $c\mathbf{x}$ belongs to H. □

PROBLEM 7.3. Consider the vector space $C(\mathbb{R})$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Let $Z : C(\mathbb{R}) \to \mathbb{R}$ be defined by $Z(f) = f(0)$.

(a) Prove that Z is a linear transformation.

Proof. We need to show that

$$Z(f+g) = Z(f) + Z(g) \quad \text{and} \quad Z(cf) = cZ(f)$$

for all $f, g \in C(\mathbb{R})$ and all $c \in \mathbb{R}$.

First, suppose $f, g \in C(\mathbb{R})$. Then

$$Z(f+g) = (f+g)(0) = f(0) + g(0) = Z(f) + Z(g).$$

Next, suppose that $f \in C(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$Z(cf) = (cf)(0) = cf(0) = cZ(f).$$

□

(b) Prove that Z is onto.

Proof. Given $x \in \mathbb{R}$, let $f_x : \mathbb{R} \to \mathbb{R}$ be the constant function with value x. Since constant functions are continuous, we see that $f_x \in C(\mathbb{R})$. Moreover, $Z(f_x) = f_x(0) = x$. □

(c) Using part (a), prove that the set $\{f \in C(\mathbb{R}) \mid f(0) = 0\}$ is a subspace of $C(\mathbb{R})$.

Proof. The set $\{f \in C(\mathbb{R}) \mid f(0) = 0\}$ is simply the kernel of Z. Since Z is a linear transformation by part (a), its kernel is a subspace of $C(\mathbb{R})$. □