Proofs Homework Set 8

MATH 217 — WINTER 2011

Due March 9

PROBLEM 8.1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be two bases of a vector space V. Prove that the coordinate vectors $\{[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}, \ldots, [\mathbf{b}_n]_\mathcal{C}\}$ form a basis of $\mathbb{R}^n$.

Proof. We need to prove that the vectors $\{[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}, \ldots, [\mathbf{b}_n]_\mathcal{C}\}$ are linearly independent and that they span $\mathbb{R}^n$.

Suppose that

$$x_1[\mathbf{b}_1]_\mathcal{C} + x_2[\mathbf{b}_2]_\mathcal{C} + \cdots + x_n[\mathbf{b}_n]_\mathcal{C} = \mathbf{0}.$$

By the linearity of the coordinate transformation $[\cdot]_\mathcal{C}$, this is the same as

$$[x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n]_\mathcal{C} = \mathbf{0}.$$

Further, because the coordinate transformation is one-to-one, this implies that

$$x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n = \mathbf{0}.$$

Since the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ form a basis of V and are thus linearly independent, this is possible only if $x_1 = x_2 = \cdots = x_n = 0$. Therefore, $\{[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}, \ldots, [\mathbf{b}_n]_\mathcal{C}\}$ is also linearly independent.

To show that the vectors $\{[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}, \ldots, [\mathbf{b}_n]_\mathcal{C}\}$ span $\mathbb{R}^n$, pick an arbitrary $\mathbf{w} \in \mathbb{R}^n$. Since the coordinate transformation $[\cdot]_\mathcal{C} : V \to \mathbb{R}^n$ is onto, there exists a vector $\mathbf{v} \in V$ such that

$$[\mathbf{v}]_\mathcal{C} = \mathbf{w}.$$

$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of V, and so these vectors span V. Thus, there exist scalars $x_1, x_2, \ldots, , x_n$ such that

$$\mathbf{v} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n.$$

Applying the coordinate transformation $[\cdot]_\mathcal{C}$ to both sides and using its linearity we obtain

$$\begin{aligned}\mathbf{w} = [\mathbf{v}]_\mathcal{C} &= [x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n]_\mathcal{C} \\ &= x_1[\mathbf{b}_1]_\mathcal{C} + x_2[\mathbf{b}_2]_\mathcal{C} + \cdots + x_n[\mathbf{b}_n]_\mathcal{C}.\end{aligned}$$

This shows that every vector $\mathbf{w} \in \mathbb{R}^n$ can be written as a linear combination of the vectors $\{[\mathbf{b}_1]_\mathcal{C}, [\mathbf{b}_2]_\mathcal{C}, \ldots, [\mathbf{b}_n]_\mathcal{C}\}$, which means these vectors span $\mathbb{R}^n$. □

PROBLEM 8.2. Let U, V, W be three vector spaces and suppose that $T : U \to V$ and $S : V \to W$ are linear isomorphisms (i.e. T and S are one-to-one and onto linear transformations). Prove that their composition $S \circ T$ is also a linear isomorphism. (Recall that the composition $S \circ T$ is the function from U to W defined by $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$ for all $\mathbf{x} \in U$. Don't forget to show that $S \circ T$ is a linear transformation!)

Proof. We need to show that the function $S \circ T$ is linear, one-to-one, and onto. We first show that $S \circ T$ is linear. Consider

$$\begin{aligned}
(S \circ T)(\mathbf{x}_1 + \mathbf{x}_2) &= S(T(\mathbf{x}_1 + \mathbf{x}_2)) \\
&= S(T(\mathbf{x}_1) + T(\mathbf{x}_2)) \\
&= S(T(\mathbf{x}_1)) + S(T(\mathbf{x}_2)) \\
&= (S \circ T)(\mathbf{x}_1) + (S \circ T)(\mathbf{x}_2), \\
(S \circ T)(a\mathbf{x}) &= S(T(a\mathbf{x})) \\
&= S(aT(\mathbf{x})) \\
&= aS(T(\mathbf{x})) \\
&= a(S \circ T)(\mathbf{x}),
\end{aligned}$$

where we have used in each middle step the fact that both S and T are linear.

To show that a linear transformation $S \circ T$ is one-to-one, it would suffice to show that the only vector $\mathbf{x} \in U$ satisfying $(S \circ T)(\mathbf{x}) = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ itself. As

$$S(T(\mathbf{x})) = \mathbf{0}$$

and S is one-to-one, we deduce that $T(\mathbf{x}) = \mathbf{0}$. Now, because T is one-to-one, this means that $\mathbf{x} = \mathbf{0}$, as required.

To show that $S \circ T$ is onto, we need to prove that the equation $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = \mathbf{y}$ has at least one solution in $\mathbf{x} \in U$ for any choice of $\mathbf{y} \in W$. Since S is onto, there is at least one $\mathbf{z} \in V$ such that

$$S(\mathbf{z}) = \mathbf{y}.$$

Since T is onto, there is at least one $\mathbf{x} \in U$ such that

$$T(\mathbf{x}) = \mathbf{z}.$$

This particular $\mathbf{x}$ will satisfy $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = S(\mathbf{z}) = \mathbf{y}$, as required. □

PROBLEM 8.3. Let V be a subspace of $\mathbb{R}^n$ with dimension $n-1$ and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ which is not in V.

(a) Show that there is a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ such that $\{\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}\}$ is a basis for V and $\mathbf{b}_n = \mathbf{x}$.

Proof. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}\}$ be any basis for V. Since $\mathbf{x} \notin V = \mathrm{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}\}$, we know from Theorem 4 in Section 4.3 that the set $\{\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}, \mathbf{x}\}$ is a linearly independent subset of $\mathbb{R}^n$. Since this is a set of size exactly n, it follows from Theorem 12 in Section 4.5 that $\{\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}, \mathbf{x}\}$ is a basis for $\mathbb{R}^n$. □

(b) Use part (a) to show that there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(\mathbf{x}) = 1$ and the kernel of T is V.

Proof. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be the basis for $\mathbb{R}^n$ that we obtained in part (a). Let $T : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation that simply returns the last $\mathcal{B}$-coordinate of a vector in $\mathbb{R}^n$. In other words, $T(\mathbf{u}) = c_n$ where

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

We first check that this is a linear transformation. This follows from the linearity of the coordinate transformation $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$. Given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

we have

$$[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix}.$$

Therefore, $T(\mathbf{u} + \mathbf{v}) = c_n + d_n = T(\mathbf{u}) + T(\mathbf{v})$. Given a vector $\mathbf{u} \in \mathbb{R}^n$ with

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

and a scalar $r \in \mathbb{R}$, we have

$$[r\mathbf{u}]_{\mathcal{B}} = r[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix}.$$

Therefore, $T(r\mathbf{v}) = rc_n = rT(\mathbf{v})$.

Now, we show that $T(\mathbf{x}) = 1$. To see this, simply note that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

since

$$\mathbf{x} = \mathbf{b}_n = 0\mathbf{b}_1 + \cdots + 0\mathbf{b}_{n-1} + 1\mathbf{b}_n.$$

Clearly, we then have $T(\mathbf{x}) = U([\mathbf{x}_{\mathcal{B}}) = 1$.

Finally, we show that $T(\mathbf{v}) = 0$ for every vector $\mathbf{v} \in V$. Since $\{\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}\}$ is a basis for V, for every vector $\mathbf{v} \in V$ we have

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \\ 0 \end{bmatrix}$$

for some scalars $c_1, \dots, c_{n-1} \in \mathbb{R}$. In other words,

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_{n-1}\mathbf{b}_{n-1} + 0\mathbf{b}_n.$$

Therefore, $T(\mathbf{v}) = 0$ for every $\mathbf{v} \in V$. $\square$