Proofs Homework Set 9

MATH 217 — WINTER 2011

Due March 16

PROBLEM 9.1.

(a) Let V be an n-dimensional vector space and let $T : V \to V$ be a linear transformation. Prove that if $\operatorname{Im}(T) = \operatorname{Ker}(T)$, then n is even.

Proof. By the Rank–Nullity Theorem, we know that

$$\dim \operatorname{Im}(T) + \dim \operatorname{Ker}(T) = n.$$

Since $\operatorname{Im}(T) = \operatorname{Ker}(T)$, we have $\dim \operatorname{Im}(T) = \dim \operatorname{Ker}(T)$ and hence

$$n = 2 \dim \operatorname{Im}(T) = 2 \dim \operatorname{Ker}(T),$$

which shows that n is even. □

(b) Give an example of such a transformation.

Proof. Let's take $V = \mathbb{R}^2$. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}.$$

Then

$$\operatorname{Im}(T) = \operatorname{Col} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and

$$\operatorname{Ker}(T) = \operatorname{Nul} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

□

PROBLEM 9.2. Let U and W be subspaces of a finite dimensional vector space V such that $U \cap W = \{\mathbf{0}\}$. Define their sum $U + W := \{u + w \mid u \in U, w \in W\}$, which is also a subspace of V. Let $\mathcal{U}$ be a basis for U and let $\mathcal{W}$ be a basis for W.

Denote $r = \dim U$ *and* $s = \dim W$. *Let* $\mathcal{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ *and* $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_s\}$.

(a) Show that $\operatorname{Span}(\mathcal{U} \cup \mathcal{W}) = U + W$.

Proof. Every vector $\mathbf{v}$ in $U + W$ can be written in the form $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u} \in U$, $\mathbf{w} \in W$. Since $\mathcal{U}$ is a basis of U, we have

$$\mathbf{u} = x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r$$

for some $x_1, \ldots, x_r$; since $\mathcal{W}$ is a basis of W, we have

$$\mathbf{w} = y_1\mathbf{w}_1 + \cdots + y_s\mathbf{w}_s$$

for some $y_1, \ldots, y_s$. Adding these equations gives

$$\mathbf{v} = \mathbf{u} + \mathbf{w} = x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r + y_1\mathbf{w}_1 + \cdots + y_s\mathbf{w}_s.$$

So, every vector $\mathbf{v}$ in $U + W$ can be written as a linear combination of the vectors in $\mathcal{U} \cup \mathcal{W}$; *i.e.* the vectors in $\mathcal{U} \cup \mathcal{W}$ span $U + W$. □

(b) Show that $\mathcal{U} \cup \mathcal{W}$ is linearly independent.

Proof. Suppose that

$$x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r + y_1\mathbf{w}_1 + \cdots + y_s\mathbf{w}_s = \mathbf{0}.$$

This can be rewritten as

$$x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r = -y_1\mathbf{w}_1 - \cdots - y_s\mathbf{w}_s.$$

Vectors on either side of this equality are the same, call them $\mathbf{v}$. Looking at the left-hand side, $\mathbf{v}$ is a linear combination of vectors in $\mathcal{U}$, so that $\mathbf{v} \in U$. Looking at the right-hand side, $\mathbf{v}$ is a linear combination of vectors in $\mathcal{W}$, so that $\mathbf{v} \in W$. This means that $\mathbf{v} \in U \cap W$, but the statement of the problem says that $U \cap W = \{\mathbf{0}\}$, so we must have $\mathbf{v} = \mathbf{0}$; *i.e.*

$$x_1\mathbf{u}_1 + \cdots + x_r\mathbf{u}_r = \mathbf{0}, \quad y_1\mathbf{w}_1 + \cdots + y_s\mathbf{w}_s = \mathbf{0}.$$

Because $\mathcal{U}$ is a basis of U, the vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ are linearly independent, so that the first of these equations is possible only if $x_1 = \cdots = x_r = 0$. Similarly, because $\mathcal{W}$ is a basis of W, the vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_s\}$ are linearly independent, and the second equation implies that $y_1 = \cdots = y_s = 0$. This shows that the only linear combination of the vectors in $\mathcal{U} \cup \mathcal{W}$ which equals the zero vector is the trivial one, so that $\mathcal{U} \cup \mathcal{W}$ is linearly independent. □

(c) Conclude from (a) and (b) that $\dim(U + W) = \dim U + \dim W$.

Proof. Parts (a) and (b) prove that $\mathcal{U} \cup \mathcal{W}$ is a basis of the vector space $U + W$. The set $\mathcal{U} \cup \mathcal{W}$ consists of $r + s$ vectors. Since all bases have the same size, we conclude that $\dim(U + W) = r + s = \dim U + \dim W$. □