Proofs Homework Set 10

MATH 217 — WINTER 2011

Due March 23

PROBLEM 10.1. Suppose that A and B are $n \times n$ matrices that commute (that is, $AB = BA$) and suppose that B has n distinct eigenvalues.

(a) Show that if $B\mathbf{v} = \lambda\mathbf{v}$ then $BA\mathbf{v} = \lambda A\mathbf{v}$.

Proof. This follows from the fact that $AB = BA$. Indeed,

$$BA\mathbf{v} = AB\mathbf{v} = A(\lambda\mathbf{v}) = \lambda A\mathbf{v}$$

since scalar multiplication commutes with matrix multiplication. □

(b) Show that every eigenvector for B is also an eigenvector for A.

Proof. Suppose $\mathbf{v}$ is an eigenvector of B with eigenvalue λ. By part (a), we have $BA\mathbf{v} = \lambda A\mathbf{v}$. So either $A\mathbf{v} = \mathbf{0}$ or $A\mathbf{v}$ is also an eigenvector of B with eigenvalue λ. Since B has n distinct eigenvalues, they all have multiplicity 1 which means that all of the eigenspaces of B are one-dimensional (see Theorem 7(b) in Section 5.3). Since $\mathbf{v}$ and $A\mathbf{v}$ both lie in the one-dimensional eigenspace of B corresponding to the eigenvalue λ, $\mathbf{v}$ and $A\mathbf{v}$ must be linearly dependent. Since $\mathbf{v} \neq \mathbf{0}$, this means that $A\mathbf{v} = \mu\mathbf{v}$ for some scalar μ. Therefore, $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue μ. □

(c) Show that the matrix A is diagonalizable.

Proof. Since B has n-distinct eigenvalues, we know that B is diagonalizable by Theorem 6 of Section 5.3. Therefore B has n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ by Theorem 5 of Section 5.3. By part (b), the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are also eigenvectors of A. Therefore, A has n linearly independent eigenvectors, which means that A is diagonalizable by Theorem 5 of Section 5.3. □

(d) Show that the matrix AB is diagonalizable.

Proof. By the solution of part (c) and Theorem 5 of Section 5.3, we have

$$A = PDP^{-1}, \qquad B = PEP^{-1}$$

where D and E are diagonal matrices and

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

Note that we get the same matrix P for A and B since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are eigenvectors of both A and B. However, the eigenvalues corresponding to these eigenvectors may be different for A and B so we get different diagonal matrices D and E.

From this, we see that

$$AB = PDP^{-1}PEP^{-1} = PDEP^{-1},$$

which shows that AB is diagonalizable since DE is a diagonal matrix. □

PROBLEM 10.2. The sequence of **Lucas numbers** is defined by the recurrence formula

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

Thus, the sequence starts like this

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

In this problem, you will use linear algebra to find an explicit formula for L_n, the nth Lucas number.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Show that $A \begin{bmatrix} L_n \\ L_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$ for each $n \geq 1$.

Proof. We have

$$A \begin{bmatrix} L_n \\ L_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L_n \\ L_{n-1} \end{bmatrix} = \begin{bmatrix} L_n + L_{n-1} \\ L_n \end{bmatrix} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$$

since $L_{n+1} = L_n + L_{n-1}$ by definition of the Lucas numbers. □

(b) Use part (a) to prove by induction that $A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$ holds for every $n \geq 0$.

Proof. Base case. First note that $A^0 = I$ by definition. Therefore, $A^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_0 \end{bmatrix}$ by definition of L_0 and L_1.

Induction step. Suppose we know that $A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$; we want to show that $A^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_{n+2} \\ L_{n+1} \end{bmatrix}$. Well, since $A^{n+1} = AA^n$, we see that

$$A^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = A \left(A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = A \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$$

by the Induction Hypothesis. But $A \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = \begin{bmatrix} L_{n+2} \\ L_{n+1} \end{bmatrix}$ by part (a), and so $A^{n+1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_{n+2} \\ L_{n+1} \end{bmatrix}$ as claimed. □

(c) Find the two eigenvalues for A. Call the positive one ϕ (this is the Greek letter "phi") and verify that the negative one is equal to $-1/\phi$.

Proof. The characteristic equation of A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

The quadratic formula gives the two roots

$$\frac{1+\sqrt{5}}{2} \qquad \frac{1-\sqrt{5}}{2}.$$

The first root is clearly positive, and so $\phi = \frac{1}{2}(1+\sqrt{5}) = 1.61803\ldots$ (This is the famous Golden Ratio!) The second root is negative, $\frac{1}{2}(1-\sqrt{5}) = -0.61803\ldots$

The fact that the second root is $-1/\phi$ can be checked by multiplying the two numbers:

$$\frac{1+\sqrt{5}}{2}\frac{1-\sqrt{5}}{2} = \frac{1-\sqrt{5}+\sqrt{5}-5}{4} = \frac{-4}{4} = -1. \quad \square$$

(d) Find eigenvectors corresponding to these two eigenvalues ϕ and $-1/\phi$.

Proof. For the first eigenvalue ϕ, we have

$$A - \phi I = \begin{bmatrix} 1-\phi & 1 \\ 1 & -\phi \end{bmatrix} \sim \begin{bmatrix} 1 & -\phi \\ 0 & 0 \end{bmatrix}$$

and thus we find the corresponding eigenvector $\begin{bmatrix} \phi \\ 1 \end{bmatrix}$.

For the second eigenvalue $-1/\phi$, we have

$$A + \frac{1}{\phi} I = \begin{bmatrix} 1+1/\phi & 1 \\ 1 & 1/\phi \end{bmatrix} \sim \begin{bmatrix} 1 & 1/\phi \\ 0 & 0 \end{bmatrix}$$

and thus we find the corresponding eigenvector $\begin{bmatrix} -1/\phi \\ 1 \end{bmatrix}$. $\square$

(e) Find an invertible matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

Proof. By Theorem 5 of Section 5.3, we must have $A = PDP^{-1}$ where

$$D = \begin{bmatrix} \phi & 0 \\ 0 & -1/\phi \end{bmatrix}, \qquad P = \begin{bmatrix} \phi & -1/\phi \\ 1 & 1 \end{bmatrix}. \quad \square$$

(f) First give an explicit formula for D^n and use this to give an explicit formula for A^n.

Proof. Powers of a diagonal matrix are easy to compute explicitly:

$$D^n = \begin{bmatrix} \phi^n & 0 \\ 0 & (-1/\phi)^n \end{bmatrix}.$$

Then we use the formula $A^n = PD^nP^{-1}$ to explicitly compute A^n as follows:

$$A^n = \begin{bmatrix} \phi & -1/\phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & (-1/\phi)^n \end{bmatrix} \left(\frac{1}{\phi + 1/\phi} \begin{bmatrix} 1 & 1/\phi \\ -1 & \phi \end{bmatrix} \right)$$

which boils down to

$$A^n = \frac{1}{\phi + 1/\phi} \begin{bmatrix} \phi^{n+1} - (-1/\phi)^{n+1} & \phi^n - (-1/\phi)^n \\ \phi^n - (-1/\phi)^n & \phi^{n-1} - (1/\phi)^{n-1} \end{bmatrix}. \quad \square$$

(g) Using parts (b) and (f), give an explicit formula for the nth Lucas number!

Proof. Using the formulas from parts (b) and (f), we obtain

$$\begin{aligned} \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \frac{1}{\phi + 1/\phi} \begin{bmatrix} \phi^{n+1} - (-1/\phi)^{n+1} & \phi^n - (-1/\phi)^n \\ \phi^n - (-1/\phi)^n & \phi^{n-1} - (1/\phi)^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{\phi + 1/\phi} \begin{bmatrix} \phi^{n+1} - (-1/\phi)^{n+1} + 2\phi^n - 2(-1/\phi)^n \\ \phi^n - (-1/\phi)^n + 2\phi^{n-1} - 2(-1/\phi)^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \phi^{n+1} + (-1/\phi)^{n+1} \\ \phi^n + (-1/\phi)^n \end{bmatrix}. \end{aligned}$$

(For the last simplification step, it really helps to notice that $\phi + 1/\phi = 1 + 2/\phi = 2\phi - 1$.) Therefore, we finally obtain the remarkable formula

$$L_n = \phi^n + (-1/\phi)^n. \quad \square$$