Proofs Homework Set 10

MATH 217 — WINTER 2011

Due March 23

PROBLEM 10.1. Suppose that A and B are $n \times n$ matrices that commute (that is, $AB = BA$) and suppose that B has n distinct eigenvalues.

(a) Show that if $B\mathbf{v} = \lambda\mathbf{v}$ then $BA\mathbf{v} = \lambda A\mathbf{v}$.

(b) Show that every eigenvector for B is also an eigenvector for A.

(c) Show that the matrix A is diagonalizable.

(d) Show that the matrix AB is diagonalizable.

PROBLEM 10.2. The sequence of **Lucas numbers** is defined by the recurrence formula

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

Thus, the sequence starts like this

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

In this problem, you will use linear algebra to find an explicit formula for L_n, the nth Lucas number.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Show that $A \begin{bmatrix} L_n \\ L_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$ for each $n \geq 1$.

(b) Use part (a) to prove by induction that $A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$ holds for every $n \geq 0$.

(c) Find the two eigenvalues for A. Call the positive one ϕ (this is the Greek letter "phi") and verify that the negative one is equal to $-1/\phi$.

(d) Find eigenvectors corresponding to these two eigenvalues ϕ and $-1/\phi$.

(e) Find an invertible matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

(f) First give an explicit formula for D^n and use this to give an explicit formula for A^n.

(g) Using parts (b) and (f), give an explicit formula for the nth Lucas number!