# Proofs Homework Set 11

MATH 217 — WINTER 2011

*Due March 30*

PROBLEM 11.1. Let $A$ be an $n \times n$ real symmetric matrix; *i.e.* all entries of $A$ are real numbers and $A^T = A$. Let $\mathbf{v} \in \mathbb{C}^n$ be an eigenvector of $A$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$, and write $\bar{\mathbf{v}} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix}$.

(Recall that the **complex conjugate** of a complex number $z = a + bi$, $a, b \in \mathbb{R}$, is the complex number $\bar{z} = a - bi$. See Appendix B of the book for properties of the complex conjugate.)

(a) Show that $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$ (so $\bar{\mathbf{v}}$ is an eigenvector of $A$ with eigenvalue $\bar{\lambda}$).

*Proof.* Taking the complex conjugate of $A\mathbf{v} = \lambda\mathbf{v}$, we obtain $A\bar{\mathbf{v}} = \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$. To see this, we repeatedly use the fact that $\overline{w+z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\bar{z}$ holds for all $w, z \in \mathbb{C}$. For the right hand side, we have

$$\overline{\lambda\mathbf{v}} = \begin{bmatrix} \overline{\lambda v_1} \\ \vdots \\ \overline{\lambda v_n} \end{bmatrix} = \begin{bmatrix} \bar{\lambda}\bar{v}_1 \\ \vdots \\ \bar{\lambda}\bar{v}_n \end{bmatrix} = \bar{\lambda}\bar{\mathbf{v}}.$$

For the left hand side, we have

$$\overline{A\mathbf{v}} = \begin{bmatrix} \overline{a_{11}v_1 + \cdots + a_{1n}v_n} \\ \vdots \\ \overline{a_{n1}v_1 + \cdots + a_{nn}v_n} \end{bmatrix} = \begin{bmatrix} \bar{a}_{11}\bar{v}_1 + \cdots + \bar{a}_{1n}\bar{v}_n \\ \vdots \\ \bar{a}_{n1}\bar{v}_1 + \cdots + \bar{a}_{nn}\bar{v}_n \end{bmatrix} = \begin{bmatrix} a_{11}\bar{v}_1 + \cdots + a_{1n}\bar{v}_n \\ \vdots \\ a_{n1}\bar{v}_1 + \cdots + a_{nn}\bar{v}_n \end{bmatrix} = A\bar{\mathbf{v}},$$

where we used the fact that all enties of $A$ are real (so $\bar{a}_{ij} = a_{ij}$). $\square$

(b) Show that $\bar{\mathbf{v}}^T A\mathbf{v} = \bar{\lambda}\bar{\mathbf{v}}^T\mathbf{v}$ and that $\bar{\mathbf{v}}^T A\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{v}$.

*Proof.* Taking the transpose of the equation $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$ from (a), we obtain

$$\bar{\mathbf{v}}^T A = \bar{\mathbf{v}}^T A^T = \bar{\lambda}\bar{\mathbf{v}}^T.$$

Multiplying on the right by $\mathbf{v}$ gives

$$\bar{\mathbf{v}}^T A\mathbf{v} = \bar{\lambda}\bar{\mathbf{v}}^T\mathbf{v}.$$

Multiplying the equation $A\mathbf{v} = \lambda\mathbf{v}$ on the left by $\bar{\mathbf{v}}^T$ gives

$$\bar{\mathbf{v}}^T A\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{v}. \quad \square$$

(c) Show that $\bar{\mathbf{v}}^T\mathbf{v} = \bar{v}_1 v_1 + \cdots + \bar{v}_n v_n$ is a positive real number.

*Proof.* Let us write $v_k = a_k + ib_k$ where $a_k, b_k \in \mathbb{R}$. Then

$$\begin{aligned}\bar{v}_1 v_1 + \cdots + \bar{v}_n v_n &= (a_1 - ib_1)(a_1 + ib_1) + \cdots + (a_n - ib_n)(a_n + ib_n) \\ &= (a_1^2 + b_1^2) + \cdots + (a_n^2 + b_n^2).\end{aligned}$$

Since a sum of squares is nonnegative, we know that the result is a nonnegative real number. To see that the sum is nonzero, we use the fact that $\mathbf{v} \neq \mathbf{0}$ (since $\mathbf{v}$ is an eigenvector). It follows that $v_k = a_k + ib_k \neq 0$ for some $k$, and hence the summand $a_k^2 + b_k^2$ cannot be zero either. $\square$

(d) Conclude that $\lambda = \bar{\lambda}$ and hence $\lambda \in \mathbb{R}$.

*Proof.* From part (b), we conclude that $\bar{\lambda}\bar{\mathbf{v}}^T\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{v}$. Since $\bar{\mathbf{v}}^T\mathbf{v}$ is a nonzero number by part (c), we can divide both sides of this equation by this number to obtain that $\bar{\lambda} = \lambda$, which is only possible when $\lambda \in \mathbb{R}$. $\square$

Therefore, the eigenvalues of a real symmetric matrix are always real numbers.

PROBLEM 11.2. For a polynomial $p(x)$ and an $n \times n$ matrix $A$, let $p(A)$ denote the matrix obtained by "plugging in" $A$ for $x$. For example, if $p(x) = x^3 - 2x^2 + 3$, then $p(A) = A^3 - 2A^2 + 3I$.

(a) Show that if $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$, prove that $p(\lambda)$ is an eigenvalue of $p(A)$.

*Proof.* For conciseness, let us write

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0.$$

Let $\mathbf{v}$ be an eigenvector of $A$ with eigenvalue $\lambda$. Consider $p(A)\mathbf{v}$. Since $A^k\mathbf{v} = \lambda^k\mathbf{v}$ for every $k$, we see that

$$\begin{aligned}p(A)\mathbf{v} &= a_m A^m\mathbf{v} + a_{m-1}A^{m-1}\mathbf{v} + \cdots + a_1 A\mathbf{v} + a_0 I\mathbf{v} \\ &= a_m\lambda^m\mathbf{v} + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda\mathbf{v} + a_0\mathbf{v} \\ &= (a_m\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0)\mathbf{v}\end{aligned}$$

Therefore, $p(A)\mathbf{v} = p(\lambda)\mathbf{v}$ which shows that $p(\lambda)$ is an eigenvalue of the matrix $p(A)$. $\square$

(b) Show that if $A$ is similar to $B$, then $p(A)$ is simlar to $p(B)$.

*Proof.* Suppose that $A = P^{-1}BP$, i.e. that $A$ is similar to $B$ via the invertible matrix $P$. As above, let us write

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0.$$

Since $A^n = P^{-1}B^nP$, we see that

$$\begin{aligned}p(A) &= a_m A^m + a_{m-1}A^{m-1} + \cdots + a_1 A + a_0 I \\ &= a_m P^{-1}B^mP + a_{m-1}P^{-1}B^{m-1}P + \cdots + a_1 P^{-1}AP + a_0 P^{-1}P.\end{aligned}$$

Factoring $P^{-1}$ on the left and factoring $P$ on the right, we obtain that

$$\begin{aligned} p(A) &= a_m P^{-1} B^m P + a_{m-1} P^{-1} B^{m-1} P + \cdots + a_1 P^{-1} A P + a_0 P^{-1} P \\ &= P^{-1}(a_m B^m + a_{m-1} B^{m-1} + \cdots + a_1 B + a_0 I)P = P^{-1} p(B) P. \end{aligned}$$

Therefore, $p(A)$ is similar to $p(B)$. □

(c) Show that if $A$ is diagonalizable and $p(\lambda)$ is the characteristic polynomial of $A$, then $p(A)$ is the zero matrix.

*Proof.* The hypotheses say that $A = P^{-1}DP$ where $P$ is some invertible matrix and $D$ is the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ being the eigenvalues of $A$. Note that these scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ are precisely the roots of the characteristic polynomial of $A$, namely

$$p(\lambda) = \det(A - \lambda I) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0.$$

By part (b), we see that $p(A)$ is similar to the matrix $p(D)$. Now we can readily compute $p(D)$ as follows:

$$\begin{aligned} p(D) &= a_n D^n + \cdots + a_1 D + a_0 I \\ &= a_n \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^n \end{bmatrix} + \cdots + a_1 \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} + a_0 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & p(\lambda_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & p(\lambda_n) \end{bmatrix}. \end{aligned}$$

Since $p(\lambda_i) = 0$ for $i = 1, 2, \ldots, n$, we see that $p(D)$ is simply the zero matrix. Therefore, $p(A) = P^{-1} p(D) P$ is the zero matrix too. □