Proofs Homework Set 11

MATH 217 — WINTER 2011

Due March 30

PROBLEM 11.1. Let A be an $n \times n$ real symmetric matrix; *i.e.* all entries of A are real numbers and $A^T = A$. Let $\mathbf{v} \in \mathbb{C}^n$ be an eigenvector of A corresponding to the eigenvalue $\lambda \in \mathbb{C}$, and write

$$\bar{\mathbf{v}} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix}.$$

(Recall that the **complex conjugate** of a complex number $z = a + bi$, $a, b \in \mathbb{R}$, is the complex number $\bar{z} = a - bi$. See Appendix B of the book for properties of the complex conjugate.)

(a) Show that $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$ (so $\bar{\mathbf{v}}$ is an eigenvector of A with eigenvalue $\bar{\lambda}$).

(b) Show that $\bar{\mathbf{v}}^T A\mathbf{v} = \bar{\lambda}\bar{\mathbf{v}}^T\mathbf{v}$ and that $\bar{\mathbf{v}}^T A\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{v}$.

(c) Show that $\bar{\mathbf{v}}^T\mathbf{v} = \bar{v}_1 v_1 + \cdots + \bar{v}_n v_n$ is a positive real number.

(d) Conclude that $\lambda = \bar{\lambda}$ and hence $\lambda \in \mathbb{R}$.

Therefore, the eigenvalues of a real symmetric matrix are always real numbers.

PROBLEM 11.2. For a polynomial $p(x)$ and an $n \times n$ matrix A, let $p(A)$ denote the matrix obtained by "plugging in" A for x. For example, if $p(x) = x^3 - 2x^2 + 3$, then $p(A) = A^3 - 2A^2 + 3I$.

(a) Show that if λ is an eigenvalue of an $n \times n$ matrix A, prove that $p(\lambda)$ is an eigenvalue of $p(A)$.

(b) Show that if A is similar to B, then $p(A)$ is simlar to $p(B)$.

(c) Show that if A is diagonalizable and $p(\lambda)$ is the characteristic polynomial of A, then $p(A)$ is the zero matrix.