Proofs Homework Set 12

MATH 217 — WINTER 2011

Due April 6

Given a vector space V, an **inner product** on V is a function that associates with each pair of vectors $\mathbf{v}, \mathbf{w} \in V$ a real number, denoted $\langle \mathbf{v}, \mathbf{w} \rangle$, satisfying the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and for all scalars $c \in \mathbb{R}$:

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(ii) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(iii) $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

(iv) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

Note that the dot product is an inner product on $\mathbb{R}^n$ by Theorem 6.1 on page 376.

PROBLEM 12.1. Let $C[0,1]$ be the space of all continuous functions $f : [0,1] \to \mathbb{R}$. Define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

for all pairs of functions f, g in $C[0,1]$. Show that this is in fact an inner product, that is, that it satisfies the four properties listed above.

PROBLEM 12.2. Whenever V is a finite-dimensional vector space with basis $\mathcal{B}$, we can use the $\mathcal{B}$-coordinate system to define an inner product on V:

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}.$$

(a) Verify that this does indeed define an inner product on V, i.e. that the four properties listed above are true.

Now consider the space $V = M_{2\times 2}$ with the standard basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(b) Let Symm be the subspace of $M_{2\times 2}$ consisting of symmetric matrices (i.e. matrices that satisfy $A = A^T$). Find the orthogonal complement $\mathsf{Symm}^{\perp}$ with respect to the inner product $\langle \bullet, \bullet \rangle_{\mathcal{B}}$.

(c) The **trace** of a matrix, denoted $\mathrm{tr}(A)$, is the sum of the entries on the main diagonal of A. Show that $\langle A, A \rangle_{\mathcal{B}} = \mathrm{tr}(A^T A)$.