Proofs Homework Set 13

MATH 217 — WINTER 2011

Due April 13

PROBLEM 13.1. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$. Recall from the previous assignment that we defined the inner product $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$.

(a) Find a matrix M such that $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{B}} = \mathbf{u}^T M \mathbf{v}$.

(b) Show that if $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is an orthonormal basis, then $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{B}} = \mathbf{u} \cdot \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

(c) Find an example to show that part (b) is not necessarily true if $\mathcal{B}$ is not an orthonormal basis.

(d) Suppose now that $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that the i, j entry of $[T]_{\mathcal{B}}$ is $\langle T(\mathbf{b}_j), \mathbf{b}_i \rangle_{\mathcal{B}}$.

PROBLEM 13.2. Let W be a subspace of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $T(\mathbf{x}) = \text{proj}_W(\mathbf{x})$.

(a) Show that for every $\mathbf{x} \in \mathbb{R}^n$, $||T(\mathbf{x})|| \leq ||\mathbf{x}||$.

(b) Show that for every $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \cdot T(\mathbf{x}) \geq 0$.

(c) Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(\mathbf{x}) = \mathbf{x} - T(\mathbf{x})$. Show that this is the orthogonal projection onto $W^\perp$.

(d) Show that $||\mathbf{x}||^2 = ||T(\mathbf{x})||^2 + ||S(\mathbf{x})||^2$.