Exercise 1.13.

This exercise is a bunch of problems about radicals of ideals. All of them are pretty easy using either definition, $r(\mathfrak{a}) = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n > 0\}$ or $r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$. I'll use the first one unless the second is way easier.

1. $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

Proof: Let $x \in \mathfrak{a}$. Then $x^1 = x \in \mathfrak{a}$, so $x \in r(\mathfrak{a})$.

2. $r(r(\mathfrak{a})) = r(\mathfrak{a})$.

Proof: Since r preserves inclusions (a trivial remark that I reserve the right to use frequently), part 1. immediately gives $r(r(\mathfrak{a})) \supseteq r(\mathfrak{a})$. For the converse, note that if $x \in r(r(\mathfrak{a}))$, then $x^n \in r(\mathfrak{a})$ for some $n > 0$, so $x^{nm} = (x^n)^m \in \mathfrak{a}$ for some $m > 0$, so $x \in r(\mathfrak{a})$.

3. $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.

Proof: We show $r(\mathfrak{a}\mathfrak{b}) \subseteq r(\mathfrak{a} \cap \mathfrak{b}) \subseteq r(\mathfrak{a}) \cap r(\mathfrak{b}) \subseteq r(\mathfrak{a}\mathfrak{b})$, which is clearly sufficient. The first $\subseteq$ is trivial from applying r to $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$. The second is trivial from applying r to $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{b}$. For the third, let $x \in r(\mathfrak{a}) \cap r(\mathfrak{b})$. Then $x^n \in \mathfrak{a}$ and $x^m \in \mathfrak{b}$ for some $n, m > 0$. It follows that $x^{n+m} = x^n x^m \in \mathfrak{a}\mathfrak{b}$, so $x \in r(\mathfrak{a}\mathfrak{b})$, as desired.

Alternatively, we could apply the second definition to 1.15.4.

4. $r(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a} = (1)$.

Proof: Recall that an ideal equals (1) if and only if it contains 1. So the claim is $1 \in \mathfrak{a} \Leftrightarrow 1 \in r(\mathfrak{a})$, which is obvious from $1^n = 1$ for all $n > 0$.

5. $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$.

Proof: For this one I'll use the second definition of r, which shows that we will have equality as long as $V(\mathfrak{a} + \mathfrak{b}) = V(r(\mathfrak{a}) + r(\mathfrak{b}))$. But this equality is trivial from the following general facts, valid for all ideals I and J: firstly, $V(I + J) = V(I) \cap V(J)$; secondly, $V(r(I)) = V(I)$. These will be verified in a later exercise.

6. If $\mathfrak{p}$ is prime, $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.

Proof: By 3. and induction we have $r(\mathfrak{p}^n) = r(\mathfrak{p})$, but $r(\mathfrak{p}) = \mathfrak{p}$ trivially from the second definition.

Problem 1.6. A ring A is such that every ideal not contained in the nilradical contains a nonzero idempotent (that is, and element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical are equal.

Proof: As always, the nilradical is contained in the Jacobson radical (every maximal ideal is prime). Suppose the Jacobson radical is not contained in the nilradical; then by hypothesis we have a nonzero idempotent e in the Jacobson radical. However, $1-e$ will then be a unit: otherwise, it generates a nontrivial ideal, which will be contained in a maximal ideal; then $1=(1-e)+e$ would be contained in the self-same maximal ideal, a contradiction. But then if we multiply the equation $(1-e)e=0$ by the inverse of $1-e$, we arrive at the contradiction $e=0$.

Problem 1.15.

1. If $\mathfrak{a}$ is the ideal generated by E, then $V(E)=V(\mathfrak{a})=V(r(\mathfrak{a}))$.

Proof: That $V(E) \supseteq V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$ is obvious from the (trivial) fact that V reverses inclusions. So it suffices to see that $V(r(\mathfrak{a})) \supseteq V(E)$. So let $\mathfrak{p} \in V(E)$. Then $\mathfrak{p}$ is an ideal containing E, hence $\mathfrak{p} \supseteq \mathfrak{a}$; applying r and using 1.13.6 gives $\mathfrak{p} \supseteq r(\mathfrak{a})$, as desired.

2. $V(0)=X$, $V(1)=\emptyset$.

Proof: Every ideal contains 0; no proper ideal (and hence no prime ideal) contains 1.

3. If $(E_i)_{i\in I}$ is a family of subsets of A, then $V(\cup E_i)=\cap V(E_i)$.

Proof: A prime ideal contains $\cup E_i$ if and only if it contains each E_i.

4. $V(\mathfrak{a}\cap\mathfrak{b})=V(\mathfrak{a}\mathfrak{b})=V(\mathfrak{a})\cup V(\mathfrak{b})$.

Proof: Since V reverses inclusions and $\mathfrak{a},\mathfrak{b} \supseteq \mathfrak{a}\cap\mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$, we clearly have $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}\cap\mathfrak{b}) \subseteq V(\mathfrak{a})\cup V(\mathfrak{b})$. Thus it suffices to show that $V(\mathfrak{a})\cup V(\mathfrak{b}) \supseteq V(\mathfrak{a}\mathfrak{b})$. We show the contrapositive. So suppose $\mathfrak{p} \notin V(\mathfrak{a})\cup V(\mathfrak{b})$. Then we can find $a\in\mathfrak{a}$ and $b\in\mathfrak{b}$ with $a,b\notin\mathfrak{p}$. Since $\mathfrak{p}$ is prime, this means $ab\notin\mathfrak{p}$; but $ab\in\mathfrak{a}\mathfrak{b}$, so $\mathfrak{p}\notin V(\mathfrak{a}\mathfrak{b})$.

Problem 1.16. Well, I sort of did this one in section. And it's hard to do in TeX.

Problem 1.17. Show that the X_f form a basis for the topology on X.

Proof: Certainly they are open: $X_f = X\setminus V(f)$. Now let $U=X\setminus V(E)$ be an arbitrary open, and $\mathfrak{p}\in U$. Then $\mathfrak{p}\not\supseteq E$, so for some $f\in E$ we have $f\notin\mathfrak{p}$, i.e. $\mathfrak{p}\in X_f$, as desired.

1. $X_f\cap X_g = X_{fg}$.

Proof: This says $f\notin\mathfrak{p}$ and $g\notin\mathfrak{p}$ if and only if $fg\notin\mathfrak{p}$. "If" follows from $\mathfrak{p}$ being an ideal, and "only if" is the definition of prime.

2. $X_f=\emptyset \Leftrightarrow f$ is nilpotent.

Proof: $X_f = \emptyset$ if and only if f is contained in every prime if and only if f is contained in the nilradical, by the second definition.

3. $X_f = X \Leftrightarrow f$ is a unit.

Proof: If f is a unit, then $(f) = 1$, so $V(f) = V((f)) = \emptyset$ and $X_f = X$. Conversely, if $X_f = X$, then no prime ideal contains f; but then (f) is forced to be trivial, since otherwise it'd be contained in a maximal ideal. So $1 \in (f)$ and f is a unit.

4. $X_f = X_g \Leftrightarrow r((f)) = r((g))$.

Proof: I claim more generally that $V(I) \subseteq V(J)$ if and only if $r(J) \subseteq r(I)$ if and only if $J \subseteq r(I)$. This is indeed more general, since applying it both ways yields that $V(I) = V(J)$ if and only if $r(I) = r(J)$. To prove the claim, note that the first condition implies the second by the second definition of r, that the second implies the third trivially from $J \subseteq r(J)$, and that the third implies the first by taking V of both sides and using 1.15.1.

Note the similarity of this whole business with the Nullstellensatz. In fact, if we define a function I from the set of subsets of $\mathrm{Spec}(A)$ to the set of ideals of A by $S \mapsto \cap_{\mathfrak{p} \in S} \mathfrak{p}$, then you can verify that I and V (which is a map going the other way) behave exactly like the I and V of the Nullstellensatz; e.g. $I(V(\mathfrak{a})) = r(\mathfrak{a})$ and $V(I(S)) = \overline{S}$.

5. X is quasi-compact.

Proof: Follows from 6. on setting $f = 1$.

6. X_f is quasi-compact for all $f \in A$.

Proof: As a lemma, let's show that $X_f \subseteq \cup_{i \in I} X_{f_i}$ if and only if $f \in r(\mathfrak{a})$, where $\mathfrak{a}$ is the ideal generated by the f_i. Indeed, $X_f \subseteq \cup_{i \in I} X_{f_i}$ if and only if $V(f) \supseteq \cap_{i \in I} V(f_i) = V(\cup_{i \in I} \{f_i\}) = V(\mathfrak{a})$, these last equalities by 1.15.1 and 1.15.3; thus this lemma follows from the general claim in the proof of 1.17.4 above.

Given the lemma, let's show the X_f are quasi-compact. It suffices to consider a covering $X_f \subseteq \cup_{i \in I} X_{f_i}$ by basic opens, as always in topology (can refine an arbitrary cover, etc.). Then the lemma shows that $f \in r(\mathfrak{a})$; but $\mathfrak{a}$ is just the set of finite linear combinations of the f_i, so we get

$$f^n = \sum_{i \in J} a_i f_i$$

for some finite subset $J \subseteq I$ and $n > 0$; this shows $f \in r(\mathfrak{a}')$ with $\mathfrak{a}'$ the ideal generated by the f_i for $i \in J$. By the lemma again we get $X_f \subseteq \cup_{i \in J} X_{f_j}$, a finite subcover.

7. An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.

Proof: This is pure point-set topology given that the X_f are quasi-compact basic opens. For "if", a cover of a finite union of X_f dudes gives a cover of each; finitely many finite subcovers give a subcover for the whole thing. For "only if", write the open subset as an (arbitrary) union of basic opens, then take a finite subcover by hypothesis.

1.18. For $x \in X$ we write x as $\mathfrak{p}_x$ when we want to think of it as a prime ideal.

1. $\{x\}$ is closed $\Leftrightarrow \mathfrak{p}_x$ is maximal.

Proof: Since $\{x\}$ is closed if and only if it equals its closure, by the next exercise it suffices to show that $\mathfrak{p}$ is maximal if and only if there are no primes strictly between it and A. "Only if" comes straight from the definition. For "if", we apply to $\mathfrak{p}$ the fact that a nontrivial ideal is always contained in a maximal ideal.

2. $\overline{\{x\}} = V(\mathfrak{p}_x)$.

Proof: Certainly $V(\mathfrak{p}_x)$ is closed and contains x. If $V(E)$ is another closed set containing x, then anything in $V(\mathfrak{p}_x)$, i.e. containing $\mathfrak{p}_x$, also contains E (transitivity of $\supseteq$); this shows $V(\mathfrak{p}_x) \subseteq V(E)$. So indeed $V(\mathfrak{p}_x) = \overline{\{x\}}$.

3. $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$.

Proof: This is a rephrasing of 2.

4. X is T_0.

Proof: Let $x \neq y$ in X. Either $\mathfrak{p}_x \not\subseteq \mathfrak{p}_y$, in which case by 3. $y \notin \overline{\{x\}}$, so with $U = X - \overline{\{x\}}$ open we have $y \in U$ and $x \notin U$; or else $\mathfrak{p}_x \not\subseteq \mathfrak{p}_y$, in which case by symmetric reasoning we have an open $V = X - \overline{\{y\}}$ with $x \in V$ and $y \notin V$.

1.19 Show that X is irreducible if and only if the nilradical of A is prime.

By 1.17.2, the nilradical is prime if and only if for all $f, g \in A$, we have that $X_{fg} = \emptyset$ implies $X_f = \emptyset$ or $X_g = \emptyset$. By 1.17.1 and contrapositive, this translates to saying that for all $f, g \in A$, we have that $X_f \neq \emptyset$ and $X_g \neq \emptyset$ implies $X_f \cap X_g \neq \emptyset$. This last condition is the same as irreducibility except that it's only for basic open sets; but this is the same thing, since any nonempty open set contains a nonempty basic open subset.