Reid 5.2. Describe the irreducible components of $V(J)$ for $J = (y^2 - x^4, x^2 - 2x^3 - x^2y + 2xy + y^2 - y)$ in $k[x, y, z]$. Here k is algebraically closed.

Answer: Note that the first generator factors as $(y-x^2)(y+x^2)$, while the second factors as $(y - x^2)(-1 - 2x + y)$. Thus $J = (y - x^2)(y + x^2, -1 - 2x + y)$, so

$$V(J) = V(y - x^2) \cup V(y + x^2, -1 - 2x + y).$$

I claim that each of $V(y - x^2)$, $V(y + x^2, -1 - 2x + y)$ is irreducible. We check the first by verifying that $(y - x^2) \subseteq k[x, y, z]$ is prime. Indeed,

$$k[x, y, z]/(y - x^2) \xrightarrow{\sim} k[x, z]$$

via $x \mapsto x$, $y \mapsto x^2$, and $z \mapsto z$, with inverse given by $x \mapsto x$, $z \mapsto z$. For the second, note that, by high school algebra,

$$V(y + x^2, -1 - 2x + y) = V(x + 1, y - 1),$$

and

$$k[x, y, z]/(x + 1, y - 1) \xrightarrow{\sim} k[z]$$

via $x \mapsto -1$, $y \mapsto 1$, $z \mapsto z$, with inverse $z \mapsto z$. Thus each of $V(y - x^2)$, $V(y+x^2, -1-2x+y) = V(x+1, y-1)$ is irreducible; to check they are the components, we just need to see that neither contains the other. Again this is high school algebra: we find that if the characteristic of k is not two, there are no containments, whereas if the characteristic is two, the second zero set is redundant, and we have just one component, the first.

Remark: The isomorphisms used to check that the above ideals are prime could also been seen geometrically in terms of the zero sets, the first as projecting the parabola-sheet to a plane, the second as recognizing that $V(x + 1, y - 1)$ is just a line. See the algebra-geometry correspondence sketched in the next exercise.

5.16. Let k be a field and $A \neq 0$ a finitely generated k-algebra. Then there exist elements $y_1, \dots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \dots, y_r]$. Moreover, if k is infinite and $x_1, \dots, x_n$ is any set of generators for A, we can choose the y_i as linear combinations of the x_j. This has the geometric consequence that if k is algebraically closed and V is a closed subvariety of k^n, then there is a linear map $k^n \to k^r$ which, when restricted to V, is [finite-to-one and] surjective.

Proof: The first part was done in lecture, so assume k infinite and let's prove the refined statement. Let's go by induction on n. The case $n = 0$ being trivial, assume $n \geq 1$. There are two cases:

First suppose that x_n is algebraic over $k[x_1, \dots, x_{n-1}]$. Then we have a nonzero polynomial in $n + 1$ variables f with $f(x_1, \dots, x_n) = 0$. Let F denote its highest degree homogeneous part of f; so $F \neq 0$. Then its dehomogenization with respect to the last variable is nonzero, hence nonzero as a function since k is infinite; thus there are $\lambda_1, \dots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$. Let $x_i' = x_i - \lambda_i x_n$. Apply the inductive hypothesis to the x_i' and get $y_1, \dots, y_r$; these are linear combinations of the x_i', hence of the x_i. I claim that A is integral over $k[y_1, \dots, y_r]$, which would mean we're done. By transitivity of integrality, it suffices to see that A is integral over $k[x_1', \dots x_{n-1}']$; since $A = k[x_1', \dots, x_{n-1}'][x_n]$, though, it suffices to see that x_n is integral over $k[x_1', \dots, x_{n-1}']$. And indeed, we have

$$f(x_1' + \lambda_1 x_n, x_2' + \lambda_2 x_n, \dots, x_{n-1}' + \lambda_{n-1} x_n, x_n) = 0,$$

which, when expanded out as a polynomial in x_n, has leading coefficient $F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$; dividing through by this we have a monic polynomial which does the job.

Now suppose that x_n is transcendental over $k[x_1, \dots, x_{n-1}]$. Apply the inductive hypothesis to $x_1, \dots, x_{n-1}$, and add x_n to the y_j that you get; this works for $x_1, \dots, x_n$.

Now for the geometric interpretation.

Lemma: Let $A \xrightarrow{\phi} B$ be a map of rings, and $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the map given by pulling back prime ideals along ϕ. Then for $\mathfrak{p} \in \mathrm{Spec}(A)$, we have a canonical bijection

$$\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) \xrightarrow{\sim} f^{-1}(\{\mathfrak{p}\}) \subseteq \mathrm{Spec}(B),$$

induced by pulling back prime ideals along the natural map $B \to B_\mathfrak{p} \to B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$.

Proof: Here by $B_\mathfrak{p}$ I mean the localization of B at the multiplicative set $\phi(A \setminus \mathfrak{p})$, and by $\mathfrak{p}B_\mathfrak{p}$ I mean $\phi(\mathfrak{p})B_\mathfrak{p}$; this is indeed an ideal, as you should check. Actually, I'll leave this whole thing for you to check; you know what the induced map on Spec is for localizations, and you know what the induced map on Spec is for modding out by an ideal; it suffices to put these two together.

Corollary: Let $A \xrightarrow{\phi} B$ be a map of rings which is *finite*, i.e. for which B is a finite A-module, and f its induced map on Spec. Then $\mathrm{im}(f) = V(\ker(\phi)) \subseteq \mathrm{Spec}(A)$. In particular, if ϕ is injective, f is surjective.

Proof: Recall that Spec of a ring is empty if and only if the ring is zero. Thus the lemma gives

$$\mathrm{im}(f) = \{\mathfrak{p} \mid B_\mathfrak{p} \neq \mathfrak{p}B_\mathfrak{p}\}.$$

However, $B_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module, so Nakayama shows that in fact

$$\mathrm{im}(f) = \{\mathfrak{p} \mid B_\mathfrak{p} \neq 0\}.$$

But $B_\mathfrak{p} = 0$ if and only if $1/1 = 0$ in $B_\mathfrak{p}$, which means by definition that there is some $s \in A \setminus \mathfrak{p}$ such that $\phi(s) = 0$, which is equivalent to saying that $\mathfrak{p} \not\supseteq \ker(\phi)$, as desired.

Remark 1: The result holds more generally if ϕ is just assumed integral; the proof then just has one tricky step. Recall that finite is the same as integral plus finitely generated, so for k-algebra maps between coordinate rings of varieties they're the same thing.

Remark 2: For a general $\phi : A \to B$, the closure of the image of f is always $V(\ker(\phi))$ (nice exercise), so the corollary is equivalent to just saying that the image of f is closed.

Remark 3: We can also show that, under the hypotheses of the corollary, the fibers of f are all finite. This is because $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, being a finite dimensional vector space over the field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, is Artinian, and thus has only finitely many prime ideals.

Remark 4: Also in this situation, if $f(\mathfrak{P}) = \mathfrak{p}$, then $\mathfrak{P}$ is maximal if and only if $\mathfrak{p}$ is. Indeed, ϕ induces an integral injection $A/\mathfrak{p} \to B/\mathfrak{P}$, from which the claim follows by a result from class. We'll probably see a more refined version of this when we get to dimension theory.

OK, back to the show. Let V be a closed subvariety of k^n; let $A = k[x_1, \ldots, x_n]/I(V)$. Use the first part of the problem to get linear forms $L_1, \ldots, L_r$ such that with $y_i = L_i(x_1, \ldots, x_n)$ the y_i are algebraically independent and A is integral (finite) over $k[y_1, \ldots, y_r]$. Define $L : k^n \to k^r$ by $L = (L_1, \ldots, L_r)$. The claim is that L restricted to V is surjective, so let $p \in k^r$. Let $ev_p : k[y_1, \ldots, y_r] \to k$ denote the evaluation at p homomorphism, and $\mathfrak{m}$ its kernel. By the corollary and the fourth remark, we can extend $\mathfrak{m}$ to a maximal ideal $\mathfrak{n}$ of A, which by the Nullstellensatz is the kernel of $ev_q : k[x_1, \ldots, x_n] \to k$ for some $q \in V$. I claim $L(q) = p$. Indeed, it suffices to trace through the y_i in

$$k[y_1, \ldots, y_r] \to A \xrightarrow{ev_q} k,$$

whose composition is ev_p (it has the same kernel as ev_p, and then there's not much choice, both being k-algebra homomorphisms).

That was ad hoc, but you can conceptualize this kind of argument in terms of an anti-equivalence of categories between affine varieties over an algebraically closed k and finitely generated reduced k-algebras.

5.18. Let k be a field, and let B be a finitely generated k-algebra. Suppose that B is a field. Then B is a finite algebraic extension of k.

Proof: First, a lemma:

Lemma: Let $A \subseteq B \subseteq C$ be rings, with A noetherian, C finitely generated over A, and C finite over B. Then B is finitely generated over A.

Proof: Suppose $C = Bc_1 + \ldots + Bc_m$, and $C = A[x_1, \ldots, x_m]$. Write out the multiplication law for C over B: $c_i c_j = \sum_k b_{ijk} c_k$, and write $x_j = \sum_i b'_{ij} c_i$. Consider $B' = A[b_{ijk}, b'_{ij}] \subseteq B$. Then C is finite over B', generated by $1, c_1, \ldots, c_m$: indeed, the set $B' + B'c_1 + \ldots + B'c_m$ is a ring since we threw in the b_{ijk}, and then it contains C since we threw in the b'_{ij}. However, B' is finitely generated over A, so by Hilbert's Basis Theorem it is noetherian; hence C, being finite, is noetherian as a B'-module, so in particular $B \subseteq C$ is finite over B', hence finitely generated over A.

Now we do the proof by induction on the number of generators. If this is zero, the statement is trivial. Otherwise let $b_1, \ldots, b_n$ generate B over k. By the inductive hypothesis, we have that B is finite over $k(b_1)$. Then the lemma implies that $k(b_1)$ is finitely generated over k. This implies that b_1 is algebraic, though:

otherwise the function field in one variable $k(x)$ is finitely generated over k, say by $f_i(x)/g_i(x)$. But by Euclid's argument there is an irreducible polynomial $p(x)$ in $k[x]$ not dividing any $g_i(x)$, and $1/p(x)$ can't lie in the algebra generated by the f_i/g_i. So b_1 is algebraic, and $k(b_1)/k$ is finite, so by transitivity so is B/k.

6.1(i) Let M be a Noetherian A-module and $u : M \to M$ a surjective module homomorphism. Then u is an isomorphism.

Proof: Forget Noetherian; this is true if M is just finite. (Recall: noetherian implies finite trivially, but the converse is true only for A noetherian). Indeed, we can argue as in Cayley-Hamilton. Let $m_1, \ldots, m_n$ generate M, and write

$$m_i = u(\sum_j a_{ij} m_j) = \sum_j a_{ij} u(m_j).$$

Consider the commutative ring $A[u] \subseteq \mathrm{End}_A(M)$, and the $n \times n$ matrix P over $A[u]$ whose ij^{th} entry is $a_{ij}u - 1$. Just as in the proof of C-H, we deduce that

$$\det(P) = 0,$$

which, when expanded out, shows that we have an equation of the form

$$a_n u^n + \ldots + a_1 u \pm 1 = 0$$

(as elements of $End_A(M)$), which plainly proves that $\ker(u) = 0$ (or plainly furnishes an inverse of u, as a polynomial in u even).

6.1 (ii) If M is Artinian and u is injective, then again u is an isomorphism.

Proof: Consider the chain $\mathrm{im}(u) \supseteq \mathrm{im}(u^2) \supseteq \ldots$; by artinianity, we have $\mathrm{im}(u^n) = \mathrm{im}(u^{n+1})$ for some n. Now let $m \in M$. By the previous equality we have $u^n(m) = u^{n+1}(m')$ for some m', which by injectivity gives $m = u(m')$, as desired.

6.5. Let X be a noetherian topological space. Then every subspace of X is noetherian, and X is quasi-compact.

Proof: Let $A \subseteq X$. Take a chain $U_1 \subseteq U_2 \subseteq \ldots$ of open subsets of A, so that we get V_i open in X with $U_i = V_i \cap A$. Consider the chain

$$V_1' \subseteq V_2' \subseteq \ldots$$

where $V_n' = \cup_{i=1}^n V_i$. This stabilizes since X is noetherian, so we have $V_n' = V_{n+k}'$ for some n and all $k \geq 0$. Intersecting with A gives $\cup_{i=1}^n U_i = \cup_{i=1}^{n+k} U_i$, which says that $U_n = U_{n+k}$ for all $k \geq 0$ since we had a chain. So every subset is noetherian.

For the second claim, let $\{U_i\}$ be an open cover of X. Consider the set of open subsets which are finite unions of U_i, and take a maximal element by noetherianity. If this maximal element weren't the whole of X, it'd miss some point, which would be in some U_i; we could add this to our union to get a contradiction.

6.6. Prove that the following are equivalent:

- X is noetherian.
- Every open subspace of X is quasi-compact.
- Every subspace of X is quasi-compact.

Proof: The first implies the third by the previous exercise, and the third trivially implies the second. So it suffices to show the second implies the first. Let $U_1 \subseteq U_2 \subseteq \ldots$ be a chain of open subsets, and let U be the union. Then the cover $\{U_i\}$ has a finite subcover, which shows that the chain stabilizes.

Problem 6.7. Prove that a noetherian space is a finite union of irreducible closed subspaces, and hence that the set of irreducible components of a noetherian space is finite.

Proof: Assume the contrary, and take, by noetherianness, a minimal element A in the set of closed subspaces which are not a finite union of irreducible closed subspaces (it is unambiguous as to whether we're saying closed in X or in A). If A is irreducible, we're cool. Otherwise we have two proper closed subspaces A' and A'' with $A' \cup A'' = A$; by minimality each of A' and A'' is a finite union of irreducible closed subspaces; thus so is A. This is a contradiction.

First we have to say what is meant by the set of "irreducible components"; let's say an irreducible component is a maximal irreducible closed subspace. I claim that if X is written as a union $\cup_i X_i$ of irreducible closed subspaces, then each irreducible component is among the X_i. Indeed, a direct translation of the definition of irreducible shows that if Y and Z are closed with $A \subseteq Y \cup Z$, we have $A \subseteq Y$ or $A \subseteq Z$. Applying this to our situation, we see that $A \subseteq X_i$ for some i, which by maximality means $A = X_i$.

The same argument, by the way, shows that if no X_i contains another (which can always be arranged by throwing some out), then the X_i are exactly the irreducible components.

6.8(a) Prove that if A is a noetherian ring, $X = \mathrm{Spec}(A)$ is a noetherian space.

Proof: Indeed, by stuff we know, there's an inclusion-reversing correspondence between closed subsets of X and radical ideals of A. The ACC for radical ideals is weaker than for all ideals.

6.8(b) Give a non-noetherian A for which $\mathrm{Spec}(A)$ is noetherian.

Example: Geometrically, we want a point with arbitrarily bad nilpotents. Let's take

$$A = k[x_1, x_2, x_3 \ldots]/\mathfrak{m}^2,$$

where $\mathfrak{m} = (x_1, x_2, \ldots)$. I claim $\mathrm{Spec}(A)$ has just one point, $\mathfrak{m}$. Indeed, this is maximal:

$$k[x_1, x_2, \ldots]/\mathfrak{m} \xrightarrow{\sim} k$$

via $x_i \mapsto 0$ for all i. But also, $\mathfrak{m}$ clearly lies inside the nilradical of A, which lies inside all primes. The only way this can work is if it's the only prime.

However, we have $(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \ldots$, which doesn't stabilize (it's pretty easy to understand operations in A: just forget all terms of degree > 1).

6.9. Deduce from Exercise 8 that the set of minimal primes in a noetherian ring is finite.

Proof: By 6.7 and 6.8, the set of irreducible components of X is finite. But maximal irreducible closed subspaces of X correspond to minimal prime ideals of A:

Indeed, recall that the operations $I \mapsto V(I)$ and $S \mapsto \cap_{\mathfrak{p} \in S} \mathfrak{p}$ give inclusion-reversing bijections between the set of radical ideals of A and the set of closed subsets of X. So all we need to check is that primality corresponds to irreducibility. But recall that the closed subset $V(I)$ is homeomorphic to $\mathrm{Spec}(A/I)$, and that by an exercise on the first problem set this is irreducible if and only if the nilradical of A/I is prime, i.e. if and only $r(I) = I$ is prime.