3.6. Let A be a nonzero ring and let Σ be the set of all multiplicative subsets S of A for which $0 \notin S$. Show that Σ has maximal elements and that $S \in \Sigma$ is maximal if and only if $A \setminus S$ is a minimal prime ideal of A.

Proof: Σ is partially-ordered by $\subseteq$; it is nonempty since $\{1\} \in \Sigma$; every chain has an upper bound (its union works, as is easily checked). Thus Zorn implies that Σ has maximal elements. Suppose $S \in \Sigma$ is maximal. Since $0 \notin S$, we have $1/0 \neq 0/0$ in $S^{-1}(A)$, so this latter ring is nonzero, and thus has a prime ideal, which corresponds to a prime ideal P of A with $A \setminus P \supseteq S$; since $A \setminus P$ is also multiplicative and also misses 0, maximality shows $A \setminus S = P$. And P must be minimal since $P \supseteq P'$ if and only if $A \setminus P' \supseteq A \setminus P$.

For the converse, if $A \setminus S$ is a minimal prime ideal, by the same Zorn's lemma argument we can find a maximal element S' of Σ containing S; by the above we have $S' = A \setminus P'$ for a prime P'; we find then $P' = A \setminus S$ by minimality, so $S = S'$ is maximal.

3.9. For $A \neq 0$ let S_0 consist of all regular elements of A (i.e. non-zero divisors). Show that S_0 is a saturated multiplicatively closed subset of A and that every minimal prime ideal of A is contained in $D = A \setminus S_0$. Prove a., b., and c. below, too.

Proof: Certainly $1 \in S_0$. So to show the first claim, we need that for $a, b \in A$, we have $ab \in S$ if and only if $a \in S$ and $b \in S$. This translates to $ab \in D$ if and only if $a \in D$ or $b \in D$. For "if", say for instance $a \in D$, i.e. there is an $a' \neq 0$ such that $aa' = 0$. Then $aba' = 0$ as well, so, yeah. For "only if", say $ab \in D$, i.e. there is a $c \neq 0$ such that $abc = 0$. If $bc = 0$ we're cool because then $b \in D$; if not we're still cool, because then $a(bc) = 0$ shows $a \in D$.

Saying every minimal prime is contained in D is the same, by the first problem, as saying that S_0 is contained in every maximal element S of Σ. But note that $S_0 \cdot S$ is a multiplicative subset of A, and it doesn't contain 0 since S_0 is the non-zero divisors and $0 \notin S$. Maximality gives $S_0 \cdot S = S$, so $S_0 \subseteq S$, as desired.

a. S_0 is the largest multiplicative subset S of A for which the map $A \to S^{-1}A$ is injective.

Proof: Let S be multiplicative. Then $A \to S^{-1}A$ is injective $\Leftrightarrow$ $a/1 = 0/0$ implies $a = 0$ $\Leftrightarrow$ $as = 0$ for some $s \in S$ implies $a = 0$ $\Leftrightarrow$ $S \subseteq S_0$.

b. Every element in $S_0^{-1}A$ is a unit or a zero-divisor.

Proof: Consider $a/s \in S_0^{-1}A$. If $a \in S_0$, we have $a/s \cdot s/a = 1$, and a/s is a unit. Otherwise $a \in D$, so there is an $a' \in A$, $a' \neq 0$, such that $aa' = 0$. Then $a/s \cdot a'/1 = 0$, and $a'/1 \neq 0$ by a. above.

c. If every element in A is a unit or zero-divisor then $A \to S_0^{-1}A$ is an isomorphism.

Proof: Then $A \xrightarrow{id} A$ trivially satisfies the same universal property as $A \to S_0^{-1}A$.

3.20. Let $f : A \to B$ be a ring homomorphism. Show the following.

a. Every prime ideal in A is a contracted ideal $\Leftrightarrow$ f^* is onto.

Proof: $\Leftarrow$ is trivial: f^* being onto is the same as every prime ideal of A being the contraction of a prime ideal. For $\Rightarrow$, suppose $P \in \operatorname{Spec}(A)$ is the contraction of $I \subseteq B$. Replacing A by A/P and B by B/I, we can assume that f is an injection (say it's an inclusion for simplicity) and $P = 0$. Let $S = A \setminus \{0\}$, a multiplicative subset of B. We have $S^{-1}B \neq 0$ since $0 \notin S$; thus $S^{-1}B$ has a prime ideal, corresponding to a prime Q of B which doesn't meet $A \setminus \{0\}$, i.e. $Q \cap A = \{0\}$, as desired.

b. Every prime ideal in B is an extended ideal $\Rightarrow$ f^* is one-to-one.

Proof: It's easy to check that for an arbitrary ideal $I \subseteq B$, we have $I^e = I^{ece}$. So suppose $P, Q \in \operatorname{Spec}(B)$ with $P^c = Q^c$. Then applying $(-)^e$ implies the result, by the hypothesis and the above.

c. Is the converse to part *b.* true?

Answer: No. Consider $k \to k[t]/t^2$ with k a field; the map on Spec is an isomoprhism, but (t) is not an extended ideal. If you want a reduced counterexample, try $k[x, y]/(y^2 - x^3) \to k[t]$ by $x \mapsto t^2$, $y \mapsto t^3$, when $k = \overline{k}$.

3.22 Show that the image of $\operatorname{Spec}(A_P)$ in $\operatorname{Spec}(A)$ is the intersection over all open neighborhoods of P.

Proof: The image consists of primes Q not meeting $A \setminus P$, i.e. primes $Q \subseteq P$. The intersection over all open neighborhoods of P is $\{Q \mid \text{for all ideals } I, \text{ we have } Q \not\supseteq I \text{ whenever } P \not\supseteq I\} = \{Q \mid \text{for all ideals } I, Q \supseteq I \text{ implies } P \supseteq I\}$, which is clearly the same thing.